Aspects of Noncommutative Gravity

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Plan of the Talk:

- Introduction
- Measurement Uncertainties at the Planck scale
- Geometry & commutative algebra
- Noncommutative geometry
- Twisted Diffeomorphisms
- Remarks

Plan of the Talk:

- Examples in 1+1 dimensions
- 2+1 gravity and quantization of time
- Interacting topologies
- Fluctuating topologies
- Concluding remarks

Introduction

- Two basic questions about nature that have been repeatedly raised in the physics literature are :
  - Can space-time coordinates be measured with arbitrary precision ?
  - Is there a fundamental and elementary length scale in nature ?
- These issues are related to the quantum structure of space-time relevant at the Planck scale.
- Noncommutative Geometry is one of the candidates for describing physics at that regime.

History - ancient

- Bohr & Rosenfeld (1933) tried to model uncertainties in measurement of space-time coordinates from fluctuations of gravitational fields.
- Heisenberg (1938) proposed the idea of an elementary length scale in particle physics.
- Markov (1940) and Snyder (1947) first introduced the idea of a quantized structure of space-time.
- De Witt (1962) discussed the modification of geometry at Planck scale due to the measurement process.

History - modern..

- More recently, in the 1980's Connes suggested that noncommutative geometry might be of interest for field theories and Woronowicz and Majid discussed the application of quantum groups in quantum field theories.
- In 1990, ambiguities in space-time measurements arose in the context of superstring theories in the work of Amati, Cialfaloni and Veneziano.
- In 1994, S. Doplicher, K. Fredenhagen and J. E. Roberts developed a more concrete model of space-time quantization induced by classical gravity.
- Similar ideas have appeared in the work of Madore, Mourad (1995) and Ashtekar (1998) also.

History - related..

- A related field goes by the name of Fuzzy Physics, proposed by J. Madore (1992), H. Grosse and P. Presnajder (1995) and developed by many others.
- It seeks to provide a finite ultraviolet cutoff preserving the symmetries of the original model.
- This is useful for numerical simulations, perhaps relevant for gravity as well.

Space-time UR

Heisenberg's Principle
+ $\Longrightarrow$ Space-time uncertainty relations
Einstein's Theory

- Measuring a space-time coordinate with an accuracy δ causes and uncertainty in the momentum $\sim \frac{1}{\delta}$.
- Neglecting rest mass, an energy of the order $\frac{1}{\delta}$ is transmitted to the system and concentrated for some time in the localization region. The associated energy-momentum tensor generates a gravitational field.
- The smaller the uncertainties in the measurement of coordinates, the stronger will be the gravitational field generated by the measurement.

Space-time UR

- To probe physics at Planck Scale l_p, the Compton wavelength $\frac{1}{M}$ of the probe must be less than l_p, hence $M > \frac{1}{l_p}$, i.e. Planck mass.
- When this field becomes so strong as to prevent light or other signals from leaving the region in question, an operational meaning can no longer be attached to the localization.
- Similarly, observations of very short time scales also require very high energies. Such observations can also form black holes and limit spatial resolutions leading to a relation of the form

$$\Delta t \Delta x \geq L^2, \quad L = \text{fundamental length}$$

Space-time UR

- Based on these arguments, Doplicher, Fredenhagen and Roberts (1994) arrived at uncertainty relations between the coordinates, which they showed could be deduced from a commutation relation of the type

$$[q_\mu, q_\nu] = iQ_{\mu\nu}$$

where q_μ are self-adjoint coordinate operators, μ, ν run over space-time coordinates and $Q_{\mu\nu}$ is an antisymmetric tensor, with the simplest possibility that it commutes with the coordinate operators.

Geometry & commutative algebra

- General Relativity describes the dynamics of space-time, assumed to be a smooth manifold. The geometry is determined by Einstein's equations.
- The geometry of a locally compact topological space X admits a dual description in terms of the commutative C^* algebra of smooth functions on X, denoted by $\mathcal{A}_0(X)$. The algebraic operations are the standard pointwise multiplication and addition of functions and the $*$ or adjoint operation is the standard complex conjugation. This algebra also has a norm compatible with the $*$ operation.

Geometry & commutative algebra

- A theorem due to Gelfand and Naimark says that every commutative C^* algebra can be written in the form $\mathcal{A}_0(X)$ for some uniquely determined space X.
- Thus mathematical structures on X, e.g. integrations, derivations, vector fields etc. can be formulated equivalently in terms of the commutative algebra $\mathcal{A}_0(X)$.
- The advantage of the algebraic formulation is that many of these mathematical structures on the C^* algebra continues to make sense even when the algebra becomes noncommutative.

Noncommutative geometry

An example of noncommutative geometry is provided by the d-dimensional Groenewold-Moyal spacetime or GM plane, which is an algebra $\mathcal{A}_\theta(\mathbb{R}^N)$ generated by elements $\hat{x}_\mu$ ($\mu \in [0, 1, 2, \cdots, N-1]$) with the commutation relation

$$[\hat{x}_\mu \,, \hat{x}_\nu] = i\theta_{\mu\nu}\mathbf{1} \,,$$

$\theta_{\mu\nu}$ being real, constant and antisymmetric in its indices. This algebra can be represented by functions of commuting variables with a twisted product

$$f * g(x) = f e^{i/2 \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g.$$

The $*$ product defines the associative but noncommutative algebra $\mathcal{A}_\theta(\mathbb{R}^N)$.

NCG

The star product provides the multiplication map in the algebra

$$\mu_\theta : \mathcal{A}_\theta(\mathbb{R}^N) \otimes \mathcal{A}_\theta(\mathbb{R}^N) \longrightarrow \mathcal{A}_\theta(\mathbb{R}^N)$$

$$\mu_\theta : f \otimes g \longrightarrow f e^{i/2 \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g \equiv f * g$$

In terms of the twist element $F_\theta = e^{i/2 \partial_\mu \otimes \theta^{\mu\nu} \partial_\nu}$

$$f * g = \mu_\theta(f \otimes g) = \mu_0[F_\theta f \otimes g]$$

where μ_0 is the pointwise multiplication map for $\theta = 0$.

- In the commutative case, we know that general relativity is a theory invariant under the symmetry group of diffeomorphisms. Assuming this continues to hold at the Planck scale where noncommutative geometry is supposed to be relevant, we must now address the issue of how diffeomorphism symmetry acts on the algebra $\mathcal{A}_\theta(\mathbb{R}^N)$.
- For that, we first discuss how a symmetry group acts on a general algebra. Then we shall go on to discuss how diffeomorphisms act on $\mathcal{A}_0(\mathbb{R}^N)$ and $\mathcal{A}_\theta(\mathbb{R}^N)$.

Symmetry on algebra

Let $\mathcal{A}$ be an algebra. $\mathcal{A}$ comes with a rule for multiplying its elements. For $f, g \in \mathcal{A}$ there exists the multiplication map μ such that

$$\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}\,,$$
$$f \otimes g \rightarrow \mu(f \otimes g)\,.$$

Now let $\mathcal{G}$ be the group of symmetries acting on $\mathcal{A}$ by a given representation $D : h \rightarrow D(\alpha)$ for $\alpha \in \mathcal{G}$. We can denote this action by

$$f \longrightarrow D(\alpha) f\,.$$

Symmetry on algebra

The action of $\mathcal{G}$ on $\mathcal{A} \otimes \mathcal{A}$ is formally implemented by the coproduct Δ

$$\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$$

The action is compatible with μ only if a certain compatibility condition between $\Delta(\alpha)$ and μ is satisfied. This action is

$$f \otimes g \longrightarrow (D \otimes D)\Delta(\alpha) f \otimes g \, ,$$

and the compatibility condition requires that

$$\mu \left((D \otimes D)\Delta(\alpha) f \otimes g \right) = D(\alpha) \, \mu(f \otimes g) \, .$$

Symmetry on algebra

The latter can be expressed neatly in terms of the following commutative diagram :

$$\begin{array}{ccc} f \otimes g & \xrightarrow{\Delta} & (D \otimes D)\Delta(\alpha) f \otimes g \\ \Big\downarrow \mu & & \Big\downarrow \mu \\ \mu(f \otimes g) & \longrightarrow & D(\alpha)\mu(f \otimes g) \end{array}$$

If a Δ satisfying the above compatibility condition exists, then $\mathcal{G}$ is an automorphism of $\mathcal{A}$. If such a Δ cannot be found, then $\mathcal{G}$ does not act on $\mathcal{A}$.

Diffeomorphism

We start by giving an algebraic description of diffeos in the commutative case. This will help us to identify the essential structures, which we shall then carry over to the NC case. Diffeos are generated by vector fields defined by

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$$

Denote the space of vector fields by V. Commutator of two vector fields $\xi, \eta \in V$ is another vector field in V given by by

$$[\xi, \eta] = \xi \times \eta = (\eta^\mu (\partial_\mu \xi^\rho) - \xi^\mu (\partial_\mu \eta^\rho)) \frac{\partial}{\partial x^\rho}$$

Diffeos

The Leibniz rule for the diffeos is given by

$$(\xi(f.g)) = (\xi f).g + f.(\xi g)$$

where $f, g \in \mathcal{A}_0(\mathbb{R}^N)$ and are multiplied by the usual commutative pointwise multiplication rule. Leibniz rule is nothing but a definition of coproduct for the diffeos, abstractly defined by

$$\Delta_0 : V \longrightarrow V \otimes V, \qquad \Delta_0(\xi) = \xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi$$

Thus the coproduct structure is essential for stating the Leibniz rule.

Diffeos

It can be shown that the coproduct satisfies the condition

$$[\Delta_0(\xi), \Delta_0(\eta)] = \Delta_0(\xi \times \eta)$$

i.e. the coproduct is compatible with the algebraic structure. This makes the space of vector fields V both and algebra and a coalgebra, or a bialgebra. Together with counit ϵ and antipode S defined by

$$\epsilon(\xi) = 0 \qquad\qquad S(\xi) = -\xi$$

V acquires the structure of a Hopf algebra.

Diffeos

Diffeos generate a coordinate transformation

$$x^{\mu} \longrightarrow x^{\mu} + \xi^{\mu}(x)$$

Under an infinitesimal coordinate transformation, the change of a scalar field ϕ is given by

$$\delta_{\xi}\phi(x) = -(\xi(\phi(x))$$

For covariant vector fields we have

$$\delta_{\xi}V_{\mu} = -\xi^{\rho}(\partial_{\rho}V_{\mu}) - (\partial_{\mu}\xi^{\rho})V_{\rho}$$

and similarly for contravariant vector fields also. This can be generalized to arbitrary tensor fields.

Diffeos

Thus we have the algebra of infinitesimal diffeos given by

$$[\delta_\xi, \delta_\eta] = \delta_{\xi \times \eta}$$

with the coproduct

$$\Delta_0 \delta_\xi = \delta_\xi \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\xi$$

This implies the important condition that

$$\mu_0\{\Delta_0(\delta_\xi) V_\mu \otimes V_\nu\} = (\delta_\xi)(V_\mu V_\nu)$$

where μ_0 denotes the usual pointwise multiplicative map and the above statement shows that the commutative diagram closes in this case.

NC Diffeos

In the noncommutative case, we have the algebra $\mathcal{A}_\theta(\mathbb{R}^N)$ with the multiplication map μ_θ. The work of Aschieri et al, based on ideas of Drinfeld shows that

- The coproduct Δ_0 is not compatible with the multiplication map μ_θ.
- One can define a new twisted coproduct

$$\Delta_\theta = F_\theta^{-1} \Delta_0 F_\theta$$

  which is compatible with μ_θ.
- This implies that the Leibniz rule is modified when $\theta \neq 0$.

NC Diffeos

- In this process, the action of any element of the diffeo group on elements of $\mathcal{A}_\theta(\mathbb{R}^N)$ is unchanged. How the diffeo acts on products of elements of $\mathcal{A}_\theta(\mathbb{R}^N)$ is modified by the twisted coproduct.
- We have so far not specified what the actual action of diffeo generators are on elements of $\mathcal{A}_\theta(\mathbb{R}^N)$. This is also not unique. We shall describe briefly two proposal, one by Wess and collaborators and another by Balachandran and collaborators, which have strikingly different physics contents.

NC Diffeo - Wess

In Wess et al's approach, the derivatives on $\mathcal{A}_\theta(\mathbb{R}^N)$ are defined by

$$\partial^*_\mu \triangleright f \equiv \partial_\mu f$$

The vector field ξ acts on $\mathcal{A}_\theta(\mathbb{R}^N)$ by the differential operator

$$X^*_\xi \triangleright f = (\xi . f) \equiv \hat{\delta}_\xi f$$

To see the Leibniz rule, note that

$$X^*_\xi \triangleright (f * g) = (\xi(f * g))$$

To see how the Leibniz rule or the coproduct can be obtained, it is useful to expand this to first order in θ.

NC Diffeo - Wess

$$
\begin{aligned}
(\xi(f * g)) &= (\xi(fg + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)(\partial_\sigma g))) \\
&= (\xi f)g + f(\xi g) + \frac{i}{2}\theta^{\rho\sigma}((\xi\partial_\rho f)(\partial_\sigma g) + (\partial_\rho f)(\xi\partial_\sigma g)) + \ldots \\
&= (\xi f) * g + f * (\xi g) \\
&\quad -\frac{i}{2}(\partial_\rho(\xi^\mu\partial_\mu f)(\partial_\sigma g) + (\partial_\rho f)(\partial_\sigma(\xi^\mu(\partial_m u g))) \\
&\quad +\frac{i}{2}((\xi^\mu(\partial_\mu\partial_\rho f))(\partial_\sigma g) + (\partial_\rho f)(\xi^\mu(\partial_\mu\partial_\sigma g))) + \ldots
\end{aligned}
$$

$$
\begin{aligned}
X_\xi^* \triangleright (f * g) &= (X_\xi^* * f) * g + f * (X_\xi^* * g) \\
&\quad \frac{-i}{2}\theta^{\rho\sigma}(([\partial_\rho^*, X_\xi^*]_* * f) * (\partial_\sigma^* * g) + (\partial_\rho^* * f) * ([\partial_\sigma^*, X_\xi^*]_* * g)) + \ldots
\end{aligned}
$$

Comultiplication rule upto first order in θ

$$
\begin{aligned}
\Delta(X_\xi^*)(f \otimes g) &= (X_\xi^* * f) \otimes g + f * (X_\xi^* \otimes g) \\
&\quad \frac{-i}{2}((X^*_{[\partial_\rho^*,\xi]} * f) \otimes (\partial_\sigma^* * g) + (\partial_\rho^* * f) \otimes ((X^*_{[\partial_\sigma^*,\xi]} * g))
\end{aligned}
$$

NC Diffeo - Wess

The Leibniz rule to all orders in θ was obtained as

$$X^*_\xi \triangleright (f * g) = \mu_\theta \{ e^{-i/2\theta^{\rho\sigma}\partial^*_\rho \otimes \partial^*_\sigma} (X^*_\xi \otimes \mathbf{1} + \mathbf{1} \otimes X^*_\xi) e^{i/2\theta^{\rho\sigma}\partial^*_\rho \otimes \partial^*_\sigma} \triangleright (f \otimes g) \}$$

The coproduct for the diffeos acting on $\mathcal{A}_\theta(\mathbb{R}^N)$ is given by

$$\Delta(\hat{\delta}_\xi) = e^{-i/2\theta^{\rho\sigma}\partial^*_\rho \otimes \partial^*_\sigma} (\hat{\delta}_\xi \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\delta}_\xi) e^{i/2\theta^{\rho\sigma}\partial^*_\rho \otimes \partial^*_\sigma}$$

With suitable definitions of counit and antipode, it can be shown that the diffeos acting on $\mathcal{A}_\theta(\mathbb{R}^N)$ has a Hopf algebra structure.

NC Poincare - Wess

- The Poincare algebra is a subalgebra of the diffeo algebra. This subalgebra in the NC case also has a Hopf algebra structure with a twisted coproduct.
- This allows to write relativistically invariant actions on $\mathcal{A}_\theta(\mathbb{R}^N)$.
- The various structures of differential geometry, including tensor calculus, metric, connection, curvature etc. can be defined in $\mathcal{A}_\theta(\mathbb{R}^N)$.
- The NC version of Einstein-Hilbert action has been constructed, which explicitly depend on the noncommutative parameter θ.

NC Poincare - Wess

The metric can be defined as

$$G_{\mu\nu} = \frac{1}{2}\left(E^a_\mu * E^b_\nu + E^a_\nu * E^b_\mu\right)\eta_{ab}$$

where $a = 1, ..4$, η_{ab} is the flat Minkowski space metric. Wess et al take the classical vierbein e^a_μ for E^a_μ. Thus $G_{\mu\nu}$ is symmetric, theta dependent and has the correct commutative limit. Inverse metric $G^{\mu\nu *}$ is defined by

$$G_{\mu\nu} * G^{\mu\rho} = \delta^\rho_\mu$$

Christoffel symbol and curvature tensors have the forms

$$\begin{aligned}
\Gamma^\sigma_{\alpha\beta} &= \frac{1}{2}\left((\partial^*_\alpha \triangleright G_{\beta\gamma} + \partial^*_\beta \triangleright G_{\alpha\gamma} - \partial^*_\gamma \triangleright G_{\alpha\beta}\right) * G^{\gamma\sigma *} \\
R^\sigma_{\mu\nu\rho} &= \partial^*_\nu \Gamma^\sigma_{\mu\rho} - \partial^*_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\beta_{\nu\rho} * \Gamma^\sigma_{\mu\beta} - \Gamma^\beta_{\mu\rho} * \Gamma^\sigma_{\nu\beta} \\
R_{\mu\nu} &= R^\sigma_{\mu\sigma\nu} \\
R &= G^{\mu\nu *} * R_{\mu\nu}
\end{aligned}$$

NC Poincare - Wess

- The action is defined as

$$S = \int d^4 x E^* * R$$

where

$$E^* = \frac{1}{4!} \epsilon^{\mu_1 \dots \mu_4}_{a_1 \dots a_4} E^{a_1}_{\mu_1} * \dots * E^{a_4}_{\mu_4}$$

- This has the correct transformation properties under diffeos, it is invariant upto total derivatives. But is is not real.
- The real Einstein-Hilbert action is defined by

$$S_{\rm EH} = \frac{1}{2} \int d^4 x \; E^* (R + \bar{R})$$

- The equations of motions can be obtained from this action, but they are not easy to solve.

NC Diffeo - Bal

For any $\alpha \in \mathcal{A}_\theta(\mathbb{R}^N)$, one can define two operators $\hat{\alpha}^{L,R}$ acting on $\mathcal{A}_\theta(\mathbb{R}^N)$:

$$\hat{\alpha}^L \xi = \alpha * \xi, \quad \hat{\alpha}^R \xi = \xi * \alpha \quad \text{for} \quad \xi \in \mathcal{A}_\theta(\mathbb{R}^N) \ ,$$

$\hat{\alpha}^{L,R}$ commute and

$$[\hat{x}^{\mu L}, \hat{x}^{\nu L}] = i\theta^{\mu\nu} = -[\hat{x}^{\mu R}, \hat{x}^{\nu R}].$$

Hence ,

$$\hat{x}^{\mu c} = \frac{1}{2}\left(\hat{x}^{\mu L} + \hat{x}^{\mu R}\right)$$

gives a representation of the commutative algebra $\mathcal{A}_0(\mathbb{R}^N)$:

$$[\hat{x}^{\mu c}, \hat{x}^{\nu c}] = 0.$$

NC Diffeo - Bal

The Lorentz group $\mathcal{L}_+^\uparrow$ (as also all diffeos) acts on functions $\alpha \in \mathcal{A}_\theta(\mathbb{R}^N)$ in just the usual way in the approach with the coproduct-twist:

$$[U(\Lambda)\alpha](x) = \alpha(\Lambda^{-1}x)$$

for $\Lambda \in \mathcal{L}_+^\uparrow$ and $U : \Lambda \rightarrow U(\Lambda)$ its representation on functions. Hence the generators $M_{\mu\nu}$ of $\mathcal{L}_+^\uparrow$ have the representatives

$$\begin{aligned} M_{\mu\nu} &= \hat{x}_\mu^c p_\nu - \hat{x}_\nu^c p_\mu, \\ p_\mu &= -i\partial_\mu \end{aligned}$$

on $\mathcal{A}_\theta(\mathbb{R}^N)$.

NC Diffeo - Bal

Vector fields v are generators of the Lie algebra of the connected component of the diffeomorphism group acting on functions. Just as for $M_{\mu\nu}$, which is a special vector field, we now see that v can be written as

$$v = v^{\mu}(\hat{x}^c)\partial_{\mu} \, .$$

Both $M_{\mu,\nu}$ and vector field look like the familiar expressions for $\theta^{\mu\nu} = 0$. Nevertheless, their action on $\mathcal{A}_{\theta}(\mathbb{R}^N)$ must involve the twisted coproduct.

NC Diffeo - Bal

We now see how Leibnitz rule is modified for $M_{\mu\nu}$. We can write, as an identity,

$$
\begin{aligned}
M_{\mu\nu}(\alpha * \beta) \quad &= \quad (M_{\mu\nu}\alpha) * \beta + \alpha * (M_{\mu\nu}\beta) \\
&\quad + \frac{1}{2}\big[(\mathrm{ad}\hat{x}_\mu \alpha) * (p_\nu \beta) - \\
&\quad (p_\nu \alpha) * (\mathrm{ad}\hat{x}_\mu \beta) - \mu \leftrightarrow \nu\big]
\end{aligned}
$$

which on using the definition of $M_{\mu\nu}$ and the antisymmetry of $\theta^{\mu\nu}$ gives

$$
\begin{aligned}
M_{\mu\nu}(\alpha * \beta) \quad &= \quad (M_{\mu\nu}\alpha * \beta) + \alpha * (M_{\mu\nu}\beta) - \\
&\quad - \frac{1}{2}\big[((p \cdot \theta)_\mu \alpha) * (p_\nu \beta) - \\
&\quad - (p_\nu \alpha) * ((p \cdot \theta)_\mu \beta) - \mu \leftrightarrow \nu\big]. \\
(p \cdot \theta)_\rho \quad &:= \quad p_\lambda \theta^\lambda_\rho .
\end{aligned}
$$

Thus the Leibnitz rule is twisted.

NC Diffeo - Bal

The twist is exactly what is required by the coproduct Δ_θ:

$$
\begin{aligned}
\Delta_\theta(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) \\
&\quad -\frac{1}{2}\Big[(p\cdot\theta)_\mu \otimes p_\nu - p_\nu \otimes (p\cdot\theta)_\mu - \mu \leftrightarrow \nu\Big] \\
\Delta_0(M_{\mu\nu}) &= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu},
\end{aligned}
$$

Thus

$$m_\theta[\Delta_\theta(M_{\mu\nu})\alpha \otimes \beta] = M_{\mu\nu}(\alpha * \beta).$$

NC Diffeo - Bal

- We conclude that The Poincare' and hence the full diffeo group is based on $\hat{x}^c$ and are isomorphic to the same groups in the $\theta_{\mu\nu} = 0$ model.
- From this we can conclude that the pure gravity action is unaffected by noncommutativity.
- This is very different from the conclusion of Wess at al and the difference lies in the action of vector fields on $\mathcal{A}_\theta(\mathbb{R}^N)$.

Remarks

- So far we have discussed the physical motivations behind considering noncommutative space-times, especially in the context of gravity.
- The diffeomorphism symmetry can be implemented in the presence of space-time noncommutativity. This is achieved by using twisted coproducts.
- With the twisted coproduct, still one has a choice of the action of diffeo generators on the GM plane and different choices lead to dramatically different physics. In Wess et al's approach, the NC gravity action depends explicitly on θ whereas in Bal et al's approach, pure gravity is unaffected by noncommutativity.

Examples in 1+1 dimensions

- Here we work essentially in the approach of Wess et al.
- Although Wess et al wrote down the Einstein-Hilbert action, it is quite complicated, and to our knowledge solutions to those equations have not yet been found.
- We shall be looking into simple in 1+1 dimensional gravity models, including cases with cosmological constant.
- These theories can be cast as gauge theories and we shall work in that formalism.
- Twisted diffeos will play a crucial role in our analysis, especially in the presence of cosmological constant.

1+1d examples

We start with a noncommutative 1 + 1 dimensional space-time and formulate two dimensional noncommutative gravity using the noncommutative version of the SO(1,1) gauge group. In the commutative case, we can think of the $SO(1,1)$ gauge group as generated by a single Pauli matrix, say σ_2. In the noncommutative case, the gauge group has two commuting generators, σ_2 and the 2×2 identity matrix I, the latter arising due to the noncommutativity of space-time.

1+1d, $\Lambda = 0$

Consider the connection one form

$$A = \omega\sigma_2 + fI$$

where ω is the spin connection and f is an additional one form. The corresponding curvature two form is

$$\begin{aligned} F &= dA + A \wedge_* A \\ &= (d\omega + f \wedge_* \omega + \omega \wedge_* f)\, \sigma_2 + (df + \omega \wedge_* \omega + f \wedge_* f)\, I \\ &\equiv F^1\sigma_2 + F^2 I\,, \end{aligned}$$

where $\wedge_*$ is understood to be the ordinary wedge product, except that the components of differential forms are now being multiplied with the Groenewold-Moyal $*$-product.

1+1d, $\Lambda = 0$

Let us also introduce a two component scalar field $\phi = (\phi_1\sigma_2 + \phi_2 I)$. Using ϕ and F we can form the gauge invariant action

$$
\begin{aligned}
S &= \frac{1}{2}\int Tr\,(\phi * F) \\
&= \frac{1}{2}\int \phi_1 * F^1 + \phi_2 * F^2 \,.
\end{aligned}
$$

S is invariant under infinitesimal gauge transformations

$$F \longrightarrow F + i[\upsilon\,, F]_*\,, \qquad \phi \longrightarrow \phi + i[\upsilon\,, \phi]_*\,,$$

where $\upsilon = \upsilon_1\sigma_2 + \upsilon_2 I$ is the gauge transformation parameter.

1+1d, $\Lambda = 0$

We can rewrite S as

$$
\begin{aligned}
S &= \frac{1}{2}\int \phi_1 * d\omega + \phi_2 * df \\
&= \frac{1}{2}\int \phi_1 \, d\omega + \phi_2 \, df \; .
\end{aligned}
$$

We thus infer that

$$d\omega = 0 = df$$

They give trivial solutions for gravity, just as in the commutative case.

1+1d, $\Lambda \neq 0$

Let us now direct our attention to the formulation of possible gauge theories with nonzero cosmological constant and with or without a dilaton. They are based on the gauge group $U(1,1) \approx SO(2,1) \times U(1)$ and its contractions. The presence of the extra $U(1)$ factor is due to the noncommutativity of the theory. The associated Lie algebra $so(2,1) \oplus u(1)$ is generated by $(Q_a, Q_2, Q_3) \equiv (P_a, J, I)\,, \quad a \in \{0,1\}$ The commutation relations among these generators are given by

$$[P_a, P_b] = -\frac{1}{2}\frac{\Lambda}{s}\varepsilon_{ab}(2J - sI)\,, \quad [P_a\,, J] = \varepsilon_a{}^b P_b\,, \quad [P_a\,, I] = [J\,, I] = 0$$

Thus I is a central element. Here Λ is the cosmological constant and s is a dimensionless parameter.

1+1d, $\Lambda \neq 0$

We use the fundamental representation

$$P_0 = \frac{1}{2}\sqrt{\frac{\Lambda}{s}} i\sigma_3 \,, \quad P_1 = \frac{1}{2}\sqrt{\frac{\Lambda}{s}} \sigma_1 \,, \quad J = \frac{1}{2}(\sigma_2 + sI) \,,$$

where $\sigma_i \,, (i = 1, 2, 3)$ denote the Pauli matrices.

Let us consider the connection one form A. It is composed of the *zweibein*'s $e^a (a = 0, 1)$, the spin connection ω and the additional one form k. Expanding in the Lie algebra basis, A reads

$$A := A^\alpha Q_\alpha = e^a P_a + \omega J + \frac{\Lambda}{2} k I \,, \quad (\alpha = 0, 1, 2, 3) \,.$$

1+1d, $\Lambda \neq 0$

The curvature associated to A is $F = dA + A \wedge_* A$, which gives

$$F = \Big[de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b) + \frac{1}{2}\Lambda(k \wedge_* e^a + e^a \wedge_* k) + \frac{s}{2}(e^a \wedge_* \omega + \omega \wedge_* e^a)\Big] P_a$$
$$+ \Big[d\omega + s\omega \wedge_* \omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b + \frac{\Lambda}{2}(k \wedge_* \omega + \omega \wedge_* k)\Big] J$$
$$+ \Big[\frac{\Lambda}{2}dk + \frac{\Lambda^2}{4}k \wedge_* k - \frac{\Lambda}{4s}h_{ab}e^a \wedge_* e^b + \frac{\Lambda}{4}\varepsilon_{ab}e^a \wedge_* e^b + \frac{1-s^2}{4}\omega \wedge_* \omega\Big] I \,.$$

Under the infinitesimal gauge transformations generated by $v = v^a P_a + v^2 J + v^3 I$, we have

$$A \longrightarrow A' = A + iD^* v \,, \quad D^* v = dv + i[v \,, A]_*$$
$$F \longrightarrow F' = F + i[v \,, F]_* \,,$$

1+1d, $\Lambda \neq 0$

We now give the gauge theory action describing NC gravity theories with nonzero cosmological constant, in two dimensions. Generalizing from the commutative theory we write,

$$S = \int Tr(\xi * F)\,,$$

Here we introduced the 4-component scalar field

$$\xi = -\frac{2s}{\Lambda}\eta^a P_a + \frac{2}{1+s^2}\eta^2 J + \frac{1}{\Lambda}\eta^3 I\,,$$

and the trace is taken over the Lie algebra basis.

1+1d, $\Lambda \neq 0$

Following the tensor calculus developed by Wess et al, the action can be shown to be invariant under twisted diffeomorphisms.

$$\delta_{\hat{\zeta}}\xi = -X^*_{\zeta}(\xi) := -\zeta^{\alpha}\partial_{\alpha}\xi$$
$$\delta_{\hat{\zeta}}(\varepsilon^{\mu\nu}F_{\mu\nu}) = -X^*_{\zeta}(\varepsilon^{\mu\nu}F_{\mu\nu}) - X^*_{(\partial_{\alpha}\zeta^{\alpha})}(\varepsilon^{\mu\nu}F_{\mu\nu}) \,.$$

Using this we find

$$\delta_{\hat{\zeta}}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) = -\partial_{\alpha}\left(\zeta^{\alpha}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu})\right) \,,$$

or equivalently

$$\delta_{\hat{\zeta}}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) = -\partial_{\alpha}\left(X^*_{\zeta^{\alpha}}(\xi * \varepsilon^{\mu\nu}F_{\mu\nu})\right) \,.$$

Thus under the infinitesimal “twisted” diffemorphisms generated by $\delta_{\hat{\zeta}}$, the Lagrangian changes by a total derivative:

$$Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) \longrightarrow Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}) - \partial_{\alpha}\left(X^*_{\zeta^{\alpha}}(Tr(\xi * \varepsilon^{\mu\nu}F_{\mu\nu}))\right) \,,$$

and hence the action S is invariant.

1+1d, $\Lambda \neq 0$

In terms of components, action S reads

$$S = \int \eta_a * \Big[de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b) + \frac{\Lambda}{2}(k \wedge_* e^a + e^a \wedge_* k) + \frac{s}{2}(e^a \wedge_* \omega + \omega \wedge_* e^a)\Big]$$
$$+ \int \eta_2 * \Big[d\omega + s\omega \wedge_* \omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b + \frac{\Lambda}{2}(k \wedge_* \omega + \omega \wedge_* k)\Big]$$
$$+ \int \eta_3 * \Big[dk + \frac{\Lambda}{2}k \wedge_* k - \frac{1}{2}h_{ab}e^a \wedge_* e^b + \frac{1}{2}\varepsilon_{ab}e^a \wedge_* e^b + \frac{1-s^2}{2\Lambda}w \wedge_* \omega\Big].$$

Recalling that one $*$-product can be removed under the integral, we get

$$\begin{aligned} S \quad = \quad & \int \eta_a * \Big(de^a + \frac{1}{2}\varepsilon_b{}^a(e^b \wedge_* \omega - \omega \wedge_* e^b)\Big) + \eta_2 * \Big(d\omega - \frac{\Lambda}{2s}\varepsilon_{ab}e^a \wedge_* e^b\Big) \\ & + \eta_3 * \Big(dk + \frac{1}{2}\varepsilon_{ab}e^a \wedge_* e^b\Big). \end{aligned}$$

1+1d, $\Lambda \neq 0$, EOM

The equations of motion following from the S when the fields η_a, η_2, η_3 are varied are given by

$$D^* e^a := de^a + \frac{1}{2}\varepsilon^a{}_b(\omega \wedge_* e^b - e^b \wedge_* \omega) = 0\,,$$
$$d\omega - \frac{\Lambda}{2s}\varepsilon_{ab} e^a \wedge_* e^b = 0\,,$$
$$dk + \frac{1}{2}\varepsilon_{ab} e^a \wedge_* e^b = 0\,.$$

They have the correct commutative limit obtained by replacing the $*$-product with the usual pointwise product, given by

$$De^a = de^a + \varepsilon^a{}_b \omega \wedge e^b = 0\,,$$
$$d\omega - \frac{1}{2}\frac{\Lambda}{s}\varepsilon_{ab} e^a \wedge e^b = 0\,,$$
$$dk + \frac{1}{2}\varepsilon_{ab} e^a \wedge e^b = 0\,.$$

We now study the solutions of the equations of motion for various values of the parameters Λ and s.

1+1d, AdS_2

The AdS_2 solution is obtained by taking both Λ and s to be finite and letting $\frac{\Lambda}{s} \rightarrow \Lambda$.
For $\Lambda < 0$, this commutative model has the well-known AdS_2 solution given by the metric

$$ds^2 = \Lambda r^2 dt^2 - \frac{1}{\Lambda r^2} dr^2$$

It gives the connection

$$A_t = \frac{i\Lambda r}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} , \quad A_r = \frac{1}{2r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The *zweibein's* and the spin connection have the form

$$e_t^0 = \sqrt{\Lambda} r \,, \quad e_t^1 = 0 \,, \quad e_r^0 = 0 \,, \quad e_r^1 = \frac{1}{\sqrt{\Lambda} r}$$
$$\omega_t = -\Lambda r \,, \quad \omega_r = 0 \,.$$

It can be easily verified that these satisfy the commutative equations of motion.

1+1d, AdS_2

- Our action is invariant under twisted diffeos. This implies

$$[x_0\,, x_1] = i\theta \quad \xrightarrow{\text{(twisted diffeos)}} \quad [t\,, r] = i\theta\,.$$

- The commutative solutions are time independent. So all the $*$ products collapse to pointwise products when they are plugged into the NC equations of motion.
- This implies that the commutative AdS_2 geometry is a solution to our noncommutative gauge theory.

1+1d, BH

In this case we keep Λ finite and take $s \to \infty$. Using the light cone coordinates

$$x^{\pm} = x^0 \pm x^1 \, .$$

the action takes the form

$$\int dx^+ dx^- \Big[\eta^+ * \big(de^+ + \frac{1}{2} (\omega \wedge_* e^- - e^- \wedge_* \omega) \big) + \eta^- * \big(de^- - \frac{1}{2} (\omega \wedge_* e^+ - e^+ \wedge_* \omega) \big) + \chi * d\omega + \eta_3 * dk + \eta_3 * e^+ \wedge_* e^- \Big] ,$$

where $\chi := \eta_2 = e^{\varphi}$ and φ is the dilaton field.

1+1d, BH

Let us first note that the variation with respect to k gives $\eta_3 =$ constant. We set this constant equal to Λ.
Variations with respect to the e^- and e^+ give

$$
\begin{aligned}
e^+ &= -\frac{1}{\Lambda} D^* \eta^+ = -\frac{1}{\Lambda}\Big(d\eta^+ - \frac{1}{2}(\eta^+ * \omega + \omega * \eta^+)\Big)\,, \\
e^- &= \frac{1}{\Lambda} D^* \eta^- = \frac{1}{\Lambda}\Big(d\eta^- + \frac{1}{2}(\eta^- * \omega + \omega * \eta^-)\Big)
\end{aligned}
$$

respectively, and from variation of ω we find

$$
d\chi = -\frac{1}{2}\{\eta^-, e^+\}_* + \frac{1}{2}\{\eta^+, e^-\}_*\,.
$$

1+1d, BH

Let us also define

$$M^* = -\Lambda\chi + \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+) \, .$$

In the commutative limit M^* approaches the black hole mass M. Using the equations of motion we find

$$dM^* = \frac{1}{4}\left[\omega \, , [\eta^+ \, , \eta^-]_*\right]_* + \frac{1}{4}\left[\eta^+ \, , [\omega \, , \eta^-]_*\right]_* \, .$$

1+1d, BH

In the commutative theory the conformally scaled metric is given by

$$\widetilde{G} = h_{ab}\frac{e^a \otimes e^b}{\chi} \equiv \frac{D\eta^+ \otimes D\eta^-}{-\frac{1}{\Lambda}(M - \eta^+\eta^-)} \; .$$

We take the ansatz below as the natural generalization of the above to the NC case:

$$\widetilde{G}^*_{\mu\nu} = \frac{1}{8}(D^*_\mu\eta^+ * D^*_\nu\eta^- + D^*_\nu\eta^+ * D^*_\mu\eta^-) * \left(\frac{1}{-\frac{1}{\Lambda}\left(M^* - \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+)\right)}\right) +$$
$$\frac{1}{8}\left(\frac{1}{-\frac{1}{\Lambda}\left(M^* - \frac{1}{2}(\eta^+ * \eta^- + \eta^- * \eta^+)\right)}\right) * (D^*_\mu\eta^+ * D^*_\nu\eta^- + D^*_\nu\eta^+ * D^*_\mu\eta^-) + (+ \longleftrightarrow -) \; .$$

We note that $\widetilde{G}_{\mu\nu}$ as given above is symmetric and transforms as a second rank covariant tensor under “twisted” diffeomorphisms. Thus according to the definition given by Wess et al, it qualifies as a metric. In what follows, we proceed by setting $\Lambda = -1$.

1+1d, BH

- We find the commutative solutions and show that they are time independent in a suitable gauge.
- The action under twisted diffeos, under which

$$[x_0\,,x_1] = i\theta \quad \xrightarrow{\text{(twisted diffeos)}} \quad [t\,,r] = i\theta\,.$$

- Using the same logic as before, we show that the commutative solutions are also solutions of the NC theory.
- Time dependent NC solutions can be obtained using suitable NC gauge transformations.

2+1 NC BH

- In the examples so far, we considered the noncommutative generalization of the commutative gravity action by replacing the pointwise products with $*$ products in the action. This was done assuming space-time noncommutativity.
- Here we want to start with a commutative solution of Einstein's equations, and then find the Poisson brackets of the space-time variables which are consistent with the geometry of the solution.
- The noncommutative gravity solutions is then obtained by representation of the algebra as operators, or by "quantization".

2+1 NC BTZ

We use our idea for the BTZ black hole, whose metric, in terms of Schwarzschild-like coordinates (r, t, ϕ) is given by

$$ds^2 = \left(M - \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2$$

$$0 \leq r < \infty \,, \; -\infty < t < \infty \,, \; 0 \leq \phi < 2\pi \,,$$

where M and J are the mass and spin, respectively, and $\Lambda = -1/\ell^2$ is the cosmological constant.

2+1 NC BTZ

For $0 < |J| < M\ell$, there are two horizons, the outer and inner horizons, corresponding respectively to $r = r_+$ and $r = r_-$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{\frac{1}{2}}\right\}$$

The two horizons coincide in the extremal case $|J| = M\ell > 0$, while the inner one disappears for $J = 0$, $M > 0$.

2+1 NC BTZ

The metric is diagonal in the coordinates (χ_+, χ_-, r), where

$$\chi_\pm = \frac{r_\pm}{\ell} t - r_\mp \phi \ ,$$

$$ds^2 = \frac{-(r^2 - r_+^2) d\chi_+^2 + (r^2 - r_-^2) d\chi_-^2}{r_+^2 - r_-^2} + \frac{\ell^2 r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} \ ,$$

which shows that

χ_+ is the time-like coordinate in the region $r \geq r_+$ (I),

r is the time-like coordinate in the region $r_- \leq r \leq r_+$ (II)

χ_- is the time-like coordinate in the region $0 \leq r \leq r_-$. (III)

2+1 NC BTZ

The manifold of the BTZ black hole solution is the quotient space of the universal covering space of AdS^3 by some elements of the group of isometries of AdS^3. The connected component of the latter is $SO(2,2)$.

Let AdS^3 is spanned by coordinates (t_1, t_2, x_1, x_2) satisfying

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 = -\ell^2$$

Alternatively, one can introduce 2×2 real matrices

$$g = \frac{1}{\ell} \begin{pmatrix} t_1 + x_1 & t_2 + x_2 \\ -t_2 + x_2 & t_1 - x_1 \end{pmatrix} \qquad \det g = 1 \ ,$$

belonging to the defining representation of $SL(2, R)$.

2+1 NC BTZ

The isometries correspond to the left and right actions on g,

$$g \longrightarrow h_L g h_R \;, \qquad h_L, h_R \in SL(2,R)$$

Since (h_L, h_R) and $(-h_L, -h_R)$ give the same action, the connected component of the isometry group for AdS^3 is

$$SL(2,R) \times SL(2,R)/Z_2 \approx SO(2,2)$$

.

2+1 NC BTZ

The BTZ black-hole is obtained by discrete identification of points on the universal covering space of AdS_3. This ensures periodicity in ϕ, $\phi \sim \phi + 2\pi$. The condition is

$$g \sim \tilde{h}_L g \tilde{h}_R \; , \qquad \tilde{h}_L, \tilde{h}_R \in SO(2,2)$$

where

$$\tilde{h}_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix} , \qquad \tilde{h}_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix}$$

Thus,

$$\mathrm{BTZ} = \frac{AdS^3}{< (\tilde{h}_L, \tilde{h}_R) >}$$

where $< (\tilde{h}_L, \tilde{h}_R) >$ denotes the group generated by $(\tilde{h}_L, \tilde{h}_R)$.

2+1 NC BTZ

For $0 < |J| < M\ell$, the universal covering space of AdS_3 is covered by three types of coordinate patches which are bounded by the two horizons at $r = r_+$ and $r = r_-$. For all three coordinate patches, g can be decomposed according to

$$g = \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_+ - \chi_-)} & 0 \\ 0 & e^{-\frac{1}{2\ell}(\chi_+ - \chi_-)} \end{pmatrix} g^{(0)}(r) \begin{pmatrix} e^{\frac{1}{2\ell}(\chi_+ + \chi_-)} & 0 \\ 0 & e^{-\frac{1}{2\ell}(\chi_+ + \chi_-)} \end{pmatrix},$$

where $g^{(0)}(r)$ is an $SO(2)$ matrix which only depends on r and the coordinate patch.

The identification breaks the $SO(2,2)$ group of isometries to a two-dimensional subgroup $\mathcal{G}_{BTZ}$, consisting of only the diagonal matrices in $\{h_L\}$ and $\{h_R\}$. $\mathcal{G}_{BTZ}$ is the isometry group of the BTZ black hole, and from is associated with translations in χ_+ and χ_-, or equivalently t and ϕ, on $r =$constant surfaces.

We shall now discuss the deformation of this solution.

2+1 NC BTZ

For generic spin, $0 < |J| < M\ell$ (and $M > 0$), we shall search for Poisson brackets for the matrix elements of g which are polynomial of lowest order. They should be consistent with the quotienting, as well as the unimodularity condition and, of course, the Jacobi identity. Writing the $SL(2,R)$ matrix as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \ ,$$

Under the quotienting, we get

$$\begin{array}{ccc} \alpha & \sim & e^{\ 2\pi r_+/\ell}\,\alpha \\ \beta & \sim & e^{-2\pi r_-/\ell}\,\beta \\ \gamma & \sim & e^{\ 2\pi r_-/\ell}\,\gamma \\ \delta & \sim & e^{-2\pi r_+/\ell}\,\delta \end{array}$$

All quadratic combinations of matrix elements scale differently, except for $\alpha\delta$ and $\beta\gamma$, which are invariant under the quotienting.

2+1 NC BTZ

Lowest order polynomial expressions for the Poisson brackets of α, β, γ and δ which are preserved under are quadratic and have the form

$$\{\alpha, \beta\} = c_1 \alpha\beta \ \ \{\alpha, \gamma\} = c_2 \alpha\gamma \ \ \{\alpha, \delta\} = f_1(\alpha\delta, \beta\gamma)$$

$$\{\beta, \delta\} = c_3 \beta\delta \{\gamma, \delta\} = c_4 \gamma\delta \{\beta, \gamma\} = f_2(\alpha\delta, \beta\gamma)$$

where c_{1-4} are constants and $f_{1,2}$ are functions.
They are constrained by

$$\begin{aligned} c_1 + c_2 &= c_3 + c_4 \\ f_1(\alpha\delta, \beta\gamma) &= (c_1 + c_2)\beta\gamma \\ f_2(\alpha\delta, \beta\gamma) &= (c_2 - c_4)\alpha\delta \ , \end{aligned}$$

after demanding that det$g$ is a Casimir of the algebra. There are three independent constants c_{1-4}.

2+1 NC BTZ

Further restrictions on the constants come from the Jacobi identity, which leads to the following two possibilities:

$$\text{A.} \quad c_2 = c_4 \qquad \text{and} \qquad \text{B.} \quad c_2 = -c_1$$

Both cases define two-parameter families of Poisson brackets. Say we call c_2 and c_3 the two independent parameters. The two cases are connected by an $SO(2,2)$ transformation.

There are three types of coordinate patches in the generic case of $M > 0$ and $0 < |J| < M\ell$, and their boundaries are the two horizons. Denote them again by: I) $r \geq r_+$, II) $r_- \leq r \leq r_+$ and III) $0 \leq r \leq r_-$.

2+1 NC BTZ

The corresponding maps to $SL(2,R)$ are given by, with

I) $r \geq r_+$,

$$g^{(0)}(r) = g_I^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & \sqrt{r^2 - r_+^2} \\ \sqrt{r^2 - r_+^2} & \sqrt{r^2 - r_-^2} \end{pmatrix},$$

II) $r_- \leq r \leq r_+$,

$$g^{(0)}(r) = g_{II}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r^2 - r_-^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & \sqrt{r^2 - r_-^2} \end{pmatrix},$$

III) $0 \leq r \leq r_-$,

$$g^{(0)}(r) = g_{III}^{(0)}(r) = \frac{1}{\sqrt{r_+^2 - r_-^2}} \begin{pmatrix} \sqrt{r_-^2 - r^2} & -\sqrt{r_+^2 - r^2} \\ \sqrt{r_+^2 - r^2} & -\sqrt{r_-^2 - r^2} \end{pmatrix},$$

2+1 NC BTZ

We can write the Poisson brackets for the various cases in terms of the Schwarzschild-like coordinates (r, t, ϕ). The results are the same in all three coordinate patches. For the two-parameter families A and B we get

A.

$$\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_+ (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_- (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2}
\end{aligned}$$

B.

$$\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_- (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_+ (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2}
\end{aligned}$$

2+1 NC BTZ

These Poisson brackets are invariant under the action of the isometry group $\mathcal{G}_{BTZ}$ of the BTZ black hole. A central element of the Poisson algebra can be constructed out of the Schwarzschild coordinates for both cases. It is given by

$$\rho_\pm = (r^2 - r_\pm^2)\exp\left\{-\frac{2\kappa\chi_\pm}{\ell}\right\}, \qquad c_2 \neq c_3 \ ,$$

where the upper and lower sign correspond to case A and B, respectively,

$$\kappa = \frac{c_3 + c_2}{c_3 - c_2} \ ,$$

The $\rho_\pm =$constant surfaces define symplectic leaves, which are topologically $\mathbb{R}^2$ for generic values of the parameters (more specifically, $c_2 \neq \pm c_3$). We can coordinatize them by χ_+ and χ_-. One then has a trivial Poisson algebra in the coordinates $(\chi_+, \chi_-, \rho_\pm)$:

$$\{\chi_+, \chi_-\} = \frac{\ell^2}{2}(c_3 - c_2) \qquad \{\rho_\pm, \chi_+\} = \{\rho_\pm, \chi_-\} = 0$$

The action of the $\mathcal{G}_{BTZ}$ transforms one symplectic leaf to another, except for the case $c_2 = -c_3$, on which we focus from now on.

2+1 NC BTZ

- $\kappa = 0$ and the radial coordinate is in the center of the algebra.
- $r =$constant define $\mathbb{R} \times S^1$ symplectic leaves, and they are invariant under the action of $\mathcal{G}_{BTZ}$.
- The coordinates ϕ and t parametrizing any such surface are canonically conjugate:

$$\{\phi, t\} = \frac{c_3 \ell^3}{r_+^2 - r_-^2} \qquad \{\phi_\pm, r\} = \{t, r\} = 0$$

2+1 NC BTZ

In passing to the noncommutative theory, we need to define a deformation of the commutative algebra generated by $t, e^{i\phi}$ and r. Call the corresponding operators $\hat{t}, e^{i\hat{\phi}}$ and $\hat{r}$, respectively. Their commutation relations are :

$$[e^{i\hat{\phi}}, \hat{t}] = \theta e^{i\hat{\phi}} \qquad [\hat{r}, \hat{t}] = [\hat{r}, e^{i\hat{\phi}}] = 0 \; ,$$

where from the constant θ is linearly related to $\ell^3/(r_+^2 - r_-^2)$. There are now two central elements in the algebra: i) $\hat{r}$ and ii) $e^{-2\pi i\hat{t}/\theta}$. From i), irreducible representations select the $\mathbb{R} \times S^1$ symplectic leaves. The action of $\mathcal{G}_{BTZ}$ preserves the irreducible representation, and in this sense we can say that the isometry of the classical solution survives.

2+1 NC BTZ

With regard to the central element ii) $e^{-2\pi i\hat{t}/\theta}$, one can identify it with $e^{i\chi}\mathbf{1}$ in an irreducible representation. The spectrum of $\hat{t}$ is then discrete

$$n\theta - \frac{\chi\theta}{2\pi}\ , \quad n \in \mathbb{Z}$$

In associating $\hat{t}$ with the Schwarzschild coordinate t, we note that the latter is the time for the exterior of the black hole, but not for the interior.

If there is a Hamiltonian description for this analysis, then the corresponding energy is conserved modulo $\frac{2\pi}{\theta}$.

Interacting topology

- Classical mechanics is formulated using the commutative algebra of functions $C^\infty(T^*Q)$ on phase space. Classical topology is encoded in this algebra.
- In the passage to quantum theory, this algebra is deformed to an appropriate noncommutative algebra. If classical topology can be identified with the commutative algebra $C^\infty(T^*Q)$, then quantum topology can be identified with its noncommutative deformation.
- The classical topology on spacetime $\mathbb{R}^{d+1}$ is coded in the commutative algebra of smooth functions $\mathcal{A}_0(\mathbb{R}^{d+1})$. The associated quantum topology is the GM plane $\mathbb{R}_\theta^{d+1}$ which is the Moyal algebra $\mathcal{A}_\theta(R^{d+1})$, with the $*$ product.

Interacting topology

- Thus, a theory where different sorts of matter and gauge fields are based on algebras $\mathcal{A}_\theta$ with different θ is a theory of interacting quantum topologies.
- This has interesting consequences for area preserving diffeomorphisms in 2 space dimensions, with applications to the cyclotron motion of electrons in a plane relevant for the quantum Hall effect.

Interacting topology

- In 2d, $\theta_{ab} = \theta\epsilon_{ab}$
- Under a transformation $x \longrightarrow x'$,

$$\epsilon_{ab} \longrightarrow \epsilon_{ab}\frac{\partial x'_a}{\partial x_c}\frac{\partial x'_b}{\partial x_d} = \det\left(\frac{\partial x'}{\partial x}\right)\epsilon_{cd}$$

- If the determinant of the transformation is 1, then the Levi-Civita tensor is invariant under the transformation. So it is invariant under area preserving diffeos. Similarly, the rotations in the plane given by $SO(2)$ also leaves the epsilon tensor in 2d invariant.

Interacting topology

Q: What is the coproduct Δ on SO(2)?

Recall that if μ_θ is the multiplication map on $\mathcal{A}_\theta(\mathbb{R}^2)$, then

$$\mu_\theta(f \otimes g) = \mu_0 \left(\mathcal{F}_\theta^{-1} f \otimes g\right) = f \star g$$

where

$$\mathcal{F}_\theta^{-1} = e^{\frac{i}{2}\partial_\mu \otimes \theta^{\mu\nu}\partial_\nu}$$

and

$$m_0(f \otimes g) = f \cdot g$$

is the pointwise multiplication in $\mathcal{A}_0(\mathbb{R}^2)$.

Interacting topology

In order for Δ to be consistent with the multiplication map on $\mathcal{A}_\theta(\mathbb{R}^2)$, it must satisfy

$$\mu_\theta\left(\Delta(\mathcal{R})(f \otimes g)\right) = \mathcal{R}(f \star g)\,.$$

Let us now note that the $SO(2)$ invariance of ε implies that any coproduct $\Delta_{\theta'}$ satisfying

$$\Delta_{\theta'}(\mathcal{R}) = \mathcal{F}_{\theta'}(\mathcal{R} \otimes \mathcal{R})\mathcal{F}_{\theta'}^{-1},$$

fulfills the above condition.

Q: So how can we fix $\theta^{'}$?

Interacting topology

A: For QHE, using the idea of interacting topologies

In quantum Hall effect, the guiding centre coordinates satisfy

$$[X_a, X_b] = i\theta\epsilon_{ab} \qquad \theta = -\frac{4}{eB}$$

where e is the electron charge and B is the external commutative magnetic field. In addition, the whole system is couple to an external commutative electric field on the 2d plane. The external electromagnetic fields are described by the abelian commutative gauge group $\mathcal{G}_c(U(1))$.

Interacting topology

If ψ, χ denote matter fields, and $e^{i\Lambda} \in \mathcal{G}_c(U(1))$, then

$$(e^{i\Lambda}\psi) * (e^{i\Lambda}\chi) \neq e^{2i\Lambda}(\psi * \chi)$$

So we must twist the coproduct on $\mathcal{G}_c(U(1))$ to

$$\Delta_\theta(e^{i\Lambda}) = \mathcal{F}_\theta^{-1}(e^{i\Lambda} \otimes e^{i\Lambda})\mathcal{F}_\theta$$

such that

$$m_\theta(\Delta_\theta(e^{i\Lambda})\psi \otimes \chi) = e^{2i\Lambda}(\psi * \chi)$$

So the coproduct on $\mathcal{G}_c(U(1))$ is fixed by its interaction with the algebra of the matter fields.

Interacting topology

We now go back to the original question of how $SO(2)$ rotations on $\mathcal{A}_\theta(\mathbb{R}^2)$. The group $SO(2)$ acts on $\mathcal{G}_c(U(1))$. Therefore, the full group that acts on the algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ is the semi-direct product

$$SO(2) \ltimes \mathcal{G}_c(U(1)) .$$

The coproduct on the rotations must preserve this group structure. Consequently, if Δ_θ is the coproduct for $\mathcal{G}_c(U(1))$, it is the same for $SO(2)$:

$$\Delta_\theta(\mathcal{R}) = \mathcal{F}_\theta(\mathcal{R} \otimes \mathcal{R})\mathcal{F}_\theta^{-1} .$$

where $R \in SO(2)$.

Interacting topology

- Thus we see that the cyclotron motion of electrons in quantum Hall effect provides a concrete example of interacting quantum topologies. In particular, the coproduct on the algebra of matter fields fixes that for the gauge fields and the diffeos.
- The application to twisted Laughlin states is under investigation.
- This could have an application to the edge states of qhe also.

Fluctuating topology

- The idea of topology change has been discussed in the context of theories of space-time by several authors, e.g. R. Sorkin and his collaborators.
- Although topology change is not permitted by classical Einstein's equations, there is no reason to believe that such restrictions must apply in any quantum theory of gravity at the Planck scale.
- In early 1990's, Balachandran discussed the idea of topology change in simple quantum models. He and his collaborators applied this idea to CFT (1993) and quantum physics (1995), which was adapted to NCG by P. Teotonio-Sobrinho, S. Vaidya et al (2003, 2004).

Fluctuating topology

- We have discussed before the result of Gelfand and Naimark whereby geometry has a dual description in terms of a commutative algebra.
- When the algebra becomes noncommutative, then a result due to A. Connes states that the information for a Riemannian manifold M, together with its metric and certain other features can be encoded in the spectral triple ($\mathcal{A}$, $\mathcal{H}$, D), where $\mathcal{A}$ is the C^* algebra of smooth functions on M, $\mathcal{H}$ is the Hilbert space of square-integrable spinors on M and D is the Dirac operator acting on $\mathcal{H}$.

Fluctuating topology

Let X denote a pair of disjoint one dimensional intervals, I_1 and I_2. They are parametrized by the coordinate $x \in [0, L]$.
Let

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in \mathcal{H}_X$$

, where ψ_1 and ψ_2 has support in I_1 and I_2 respectively.
The Hilbert space has a scalar product

$$(\psi, \chi) = \int_0^L dx (\psi_1^* \chi_1 + \psi_2^* \chi_2)(x)$$

Fluctuating topology

The Dirac operator has the form

$$D = \begin{pmatrix} -i\partial_x & 0 \\ 0 & -i\partial_x \end{pmatrix}$$

However, D is an unbounded differential operator acting on the infinite dimensional Hilbert space $\mathcal{H}_X$. We have to therefore specify the boundary conditions in order to describe it completely. This follows from the von Neumann's theory of self-adjoint extensions. We shall see that the choice of topology is coded in the boundary conditions, i.e. the Dirac operator.

Fluctuating topology

For the momentum operator in a single line interval, the allowed boundary condition which makes it self-adjoint are given by

$$\psi(x = L) = e^{i\alpha}\psi(x = 0), \quad \alpha \in \mathbf{R} \bmod 2\pi$$

So this operator admits a 1-parameter family of self-adjoint extensions, parametrized by the $U(1)$ worth of boundary conditions given by $e^{i\alpha}$.
Similarly for the Dirac operator D, defined on a pair of line intervals, the corresponding self-adjoint extensions are parametrized by an $U(2)$ matrix.

Fluctuating topology

The corresponding boundary condition on our $\psi(x)$ is

$$\begin{pmatrix} \psi_1(L) \\ \psi_2(L) \end{pmatrix} = g \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix}$$

where $g \in U(2)$. We can choose for example,

$$g_1 = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \quad \text{or} \quad g_2 = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{i\beta} & 0 \end{pmatrix}$$

For the first choice, we have

$$(\psi_1^* \chi_1)(L) = (\psi_1^* \chi_1)(0) \qquad (\psi_2^* \chi_2)(L) = (\psi_2^* \chi_2)(0)$$

which indicates the topology of two disjoint circles. For the second choice, we have,

$$(\psi_1^* \chi_1)(L) = (\psi_2^* \chi_2)(0) \qquad (\psi_2^* \chi_2)(L) = (\psi_1^* \chi_1)(0)$$

indicating the topology of a single circle obtained from the two line intervals.

Fluctuating topology

- This simple example illustrates the possibility of having different topologies arising from the choice on boundary conditions for the Dirac operator.
- The connection to geometry comes through its dual description following Connes idea of the spectral triple.
- Although this example uses commutative algebra, similar ideas may be possible for noncommutative algebras as well.

Concluding remarks

- We have seen that noncommutative geometry finds a natural setting in formulating a quantum theory of gravity.
- Following the ground breaking work of Wess at al, a lot of new directions have opened up in this field. However, not many exact solutions have been found yet.
- An interesting idea to follow could be that of noncommutative holography. This has already been discussed by Manin et al. (2002) who discussed bulk/boundary correspondence when the boundary is a noncommutative space. If both NCG and holography are relevant features of Planck scale physics, then they must at least be compatible.

Concluding remarks

- Applications of NC Gravity to cosmology could be another interesting area.
- Lot of exciting work remains to be done.