Space-time Noncommutativity and quantised evolution

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noncommutative spacetime & unitary ...

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$$[\,\hat{x}_\mu, \hat{x}_\nu\,] \;=\; i\theta\epsilon_{\mu\nu}\mathcal{I}$$

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- Its usually remarked that this leads to non unitary quantum theory. We believe this is due to incorrect appreciation of the role of "Time".
- But the correct statement is if a group of transformations cannot be implemented on the algebra $\mathcal{A}_\theta\left(\mathcal{R}^2\right)$ generated by $\hat{x}_\mu$ with our relation then it will not be a symmetry Even this should be improved - will come back later

noncommutative spacetime ...

◇ We readily see that spacetime translations are automorphisms of $\mathcal{A}_\theta\left(\mathcal{R}^2\right)$: With $\mathcal{U}(\vec{a})\hat{x}_\mu = \hat{x}_\mu + a_\mu$ we see that,

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- Without the time-translation automorphism, we cannot formulate conventional quantum physics.

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- Causality: It is impossible to localize (the representation of) “coordinate” time $\hat{x}_0$ in $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ sharply. This leads to failure of causality Chaichian et al.

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- The following important point was emphasised to us by Doplicher. In quantum mechanics, if $\hat{p}$ is momentum, $\exp(i\xi\hat{p})$ is spatial translation by amount ξ. This ξ is not the eigenvalue of the position operator $\hat{x}$. In the same way, the amount τ of time translation is not "coordinate time", the eigenvalue of $\hat{x}_0$. It makes sense to talk about a state and its translate by $U(\tau)$

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- Concepts like duration of an experiment for $\theta = 0$ are expressed using $U(\tau)$. They carry over to the noncommutative case too.

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Representation theory..

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- To each $\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)$, we associate its left and right regular representations $\hat{\alpha}^L$ and $\hat{\alpha}^R$,

$$\hat{\alpha}^L \hat{\beta} = \hat{\alpha}\hat{\beta}\ ,\ \hat{\alpha}^R \hat{\beta} = \hat{\beta}\hat{\alpha}\ ,\ \hat{\beta} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)\ ,$$

with $\hat{\alpha}^L \hat{\beta}^L = \left(\hat{\alpha}\hat{\beta}\right)^L$ and $\hat{\alpha}^R \hat{\beta}^R = \left(\hat{\beta}\hat{\alpha}\right)^R$. The carrier space of this representation is $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ itself.

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- An “inner”product on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$ is needed for an eventual construction of a Hilbert space.

Representation theory..

- Consider a map $\chi : \mathcal{A}_\theta\left(\mathbb{R}^2\right) \to \mathbb{C}$ which is also positive,i.e.,

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- We illustrate these ideas briefly in the context of the commutative case, when $\theta = 0$

The commutative case

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- There is a family of positive maps χ_t of interest obtained by integrating i ψ in x_1 at “time” t:

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- We get a family of spaces $\mathcal{C}_t$ with a positive-definite sesquilinear form $(. \, , \, .)_t$:

$$\left(\psi, \varphi\right)_t = \int dx_1 \, \psi^*(t, x_1)\varphi(t, x_1) \ .$$

The commutative case

- ◇ Every function $\hat{\alpha}$ which vanishes at time t is a two-sided ideal $\mathcal{I}_t^{\theta=0} = \mathcal{I}_t^0$ of $\mathcal{C}$. As elements of $\mathcal{C}_t$, they become null vectors.

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- The completion $\overline{\mathcal{C}_t/\mathcal{N}_t^0}$ of $\mathcal{C}_t/\mathcal{N}_t^0$ in this scalar product gives a Hilbert space $\widehat{\mathcal{H}}_t^0$

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- For elements $\psi + \mathcal{N}_t^0$ and $\chi + \mathcal{N}_t^0$ in $\mathcal{C}_t/\mathcal{N}_t^0$, the scalar product is

$$\left(\psi + \mathcal{N}_t^0, \chi + \mathcal{N}_t^0\right)_t = (\psi, \chi)_t \ .$$

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- Then χ_t is a good scalar product on $\tilde{\mathcal{H}}_{0,t}$ and the quantum mechanical Hilbert space is given by $\mathcal{H}_t^0 = \overline{\tilde{\mathcal{H}}_{0,t}}$, the completion of $\tilde{\mathcal{H}}_{0,t}$.

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- The subspace $\tilde{\mathcal{H}}_{0,t}$ depends on the Hamiltonian H and is chosen as follows.

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$$\tilde{\mathcal{H}}_{0,t} = \{\psi \in \mathcal{C}_t : (i\partial_{x_0} - H)\,\psi(x_0, x_1) = 0\} \quad .$$

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- The operator $i\partial_{x_0}$ is not hermitian on all of $\mathcal{C}_t$:

$$(\psi, i\partial_{x_0}\chi)_t \neq (i\partial_{x_0}\psi, \chi)_t \ \text{ for generic } \psi, \chi \in \mathcal{C}_t \ ,$$

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$$(\psi, i\partial_{x_0}\chi)_t = (i\partial_{x_0}\psi, \chi)_t \quad \text{for generic } \psi, \chi \in \tilde{\mathcal{H}}_{0,t} \quad ,$$

- but on $\tilde{\mathcal{H}}_{0,t}$, it fulfills this property:

The commutative case

- We notice since,

$$\psi(x_0 + \tau, x_1) = \left(e^{-i\tau(i\partial_{x_0})}\psi\right)(x_0, x_1)$$

$$= \left(e^{-i\tau H}\psi\right)(x_0, x_1) \ ,$$

time evolution preserves the norm of $\psi \in \tilde{\mathcal{H}}_{0,t}$. Therefore if it vanishes at $x_0 = t$, it vanishes identically and is the zero element of $\tilde{\mathcal{H}}_{0,t}$: the only null vector in $\tilde{\mathcal{H}}_{0,t}$ is 0:

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- The completion of $\tilde{\mathcal{H}}_{0,t}$ is the quantum Hilbert space $\mathcal{H}_t^0$. There is no convenient inclusion of $\mathcal{H}_t^0$ in $\widehat{\mathcal{H}}_t^0$.

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- An observable $\hat{K}$ has to respect the Schrödinger constraint and leave $\tilde{\mathcal{H}}_{0,t}$ (and hence $\mathcal{H}_t^0$) invariant. This means that

$$\left[i\partial_{x_0} - H, \hat{K}\right] = 0 \ .$$

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- Under time translation, $\hat{x}_0$ in $\hat{K}$ shifts to $\hat{x}_0 + \tau$ as it should:

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- What we have described above leads to conventional physics. As expected $\hat{x}_0$ is not an observable as it does not commute with $i\partial_{x_0} - H$:

$$[\hat{x}_0, i\partial_{x_0} - H] = -i\mathbb{I} \ .$$

The noncommutative Case

- The above discussion shows that for quantum theory, what we need are: (1) a suitable inner product on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$; (2) a Schrödinger constraint on $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$; and (3) a Hamiltonian $\hat{H}$ and observables which act on the constrained subspace of $\mathcal{A}_\theta\left(\mathbb{R}^2\right)$.

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- We also require that (1) is compatible with the self-adjointness of $\hat{H}$ and classically real observables.
- We now consider these one by one.

The symbol calculus

- The first inner product is based on symbol calculus. If $\hat{\alpha} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right)$, we write it as

$$\hat{\alpha} = \int d^2k\, \tilde{\alpha}(k) e^{ik_1\hat{x}_1} e^{ik_0\hat{x}_0} \ ,$$

and associate the symbol α_S with $\hat{\alpha}$

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- The symbol is a function on $\mathbb{R}^2$. It is NOT the MOYAL symbol. Using this symbol, we can define a positive map S_t by

$$S_t\left(\hat{\alpha}\right) = \int dx_1\, \alpha_S(t, x_1) \ .$$

The Schrödinger constraint

◇ The noncommutative analogue "$i\frac{\partial}{\partial x_0}$" is

$$i\frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta}\mathrm{ad}\,\hat{x}_1 \ ,$$

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◇ We can write Hamiltonian as $\hat{H} = \hat{H}\left(\hat{x}_1^L, \hat{P}_1\right) \ .$

The Schrödinger constraint

◇ The noncommutative analogue "$i\frac{\partial}{\partial x_0}$" is

$$i\frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta}\mathrm{ad}\,\hat{x}_1 \ ,$$

◇ If the Hamiltonian $\hat{H}$ is time-independent,

$$[i\partial_{x_0}, \hat{H}] = 0$$

◇ We can write Hamiltonian as $\hat{H} = \hat{H}\left(\hat{x}_1^L, \hat{P}_1\right)$.

◇ If $\hat{H}$ has time-dependence then • is not correct, it will have $\hat{x}_0^L$, $\hat{x}_0^R$. But $\hat{x}_0^L = \theta\hat{P}_1 + \hat{x}_1^R$, so in the time-dependent case we write $\hat{H} = \hat{H}(\hat{x}_0^R, \hat{x}_1^L, \hat{P}_1)$

The Schrödinger constraint

- The states constrained by the Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta = \left\{\hat{\psi} \in \mathcal{A}_\theta\left(\mathbb{R}^2\right) : \left(i\partial_{x_0} - \hat{H}\right)\hat{\psi} = 0\right\} \ ,$$

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- The solutions are easy to construct:

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \Longrightarrow \hat{\psi} = e^{-i\left(\hat{x}_0^R - \tau_I\right)\hat{H}\left(\hat{P}_1, \hat{x}_1^L\right)} \hat{\chi}\left(\hat{x}_1\right)$$

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$$U\left(\hat{x}_0^R, \tau_I\right) = T \exp\left[-i \left(\int_{\tau_I}^{x_0} d\tau \, \hat{H}\left(\tau, \hat{x}_1^L, \hat{P}_1\right) \right) \right] \Bigg|_{x_0 = \hat{x}_0^R}$$

Some observations

◇ An alternative useful form for $\hat{\psi}$ is

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  where the integral can be defined at the lower limit using the usual adiabatic cut-off.

- The Hilbert spaces $\mathcal{H}_\theta^S$ and $\mathcal{H}_\theta^V$ based on scalar products $(.,.)_S$ and $(.,.)_V$ are obtained from $\tilde{\mathcal{H}}_\theta$ by completion. Our basic assumption is that $\hat{H}$ is self-adjoint in the chosen scalar product.

Some observations

◇ In the passage from H to $\hat{H}$, there is an apparent ambiguity. We replaced x_0 by $\hat{x}_0^L$, but we may be tempted to replace x_0 by $\hat{x}_0^R$. But it is incorrect to replace x_0 by $\hat{x}_0^R$ and at the same time x_1 by $\hat{x}_1^L$. Time and space should NOT commute when θ becomes nonzero whereas $\hat{x}_0^R$ and $\hat{x}_1^L$ commute.

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- Note that $\hat{x}_0^L = -\theta \hat{P}_1 + \hat{x}_0^R$ and that $\hat{x}_0^R$ behaves much like the $\theta = 0$ time x_0. *Thus if H has time-dependence, its effect on $\hat{H}$ is to induce new momentum-dependent terms* leading to nonlocal ("acausal") interactions.

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- We can construct observables as before and no complications are encountered.

A spectral map:

◇ For $\theta = 0$ let the Hamiltonian be: $H = -\frac{1}{2m}\frac{\partial^2}{\partial x_1^2} + V(\hat{x}_1)$ with eigenstates ψ_E fulfilling the Schrödinger constraint:

$$\psi_E\,(\hat{x}_0, \hat{x}_1) = \varphi_E(\hat{x}_1)e^{-iE\hat{x}_0}\;, H\varphi_E = E\varphi_E\;.$$

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$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1) \;.$$

- Then $\hat{H}$ has exactly the same spectrum as H and its eigenstates $\hat{\psi}_E$ are obtained from ψ_E.

$$\hat{\psi}_E = \varphi_E(\hat{x}_1)e^{-iE\hat{x}_0} \;, \hat{H}\varphi_E(\hat{x}_1) = E\varphi_E(\hat{x}_1) \;.$$

QFT.....:

- We can also see how to do perturbative qft's, our approach can be inferred from the work of Doplicher et al. We require of $\hat{\Phi}$ that it is a solution of the massive Klein-Gordon equation: $\left(\mathrm{ad}\hat{P}_0^2 - \mathrm{ad}\hat{P}_1^2 + \mu^2\right)\hat{\Phi} = 0$.

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- The plane wave solutions are $\hat{\phi}_k = e^{ik\hat{x}_1}e^{-i\omega(k)\hat{x}_0}$, $\omega(k)^2 - k^2 = \mu^2$.
- So for $\hat{\Phi}$, we write:

$$\hat{\Phi} = \int \frac{dk}{2\omega(k)} \left[a_k\hat{\phi}_k + a_k^\dagger\hat{\phi}_k^\dagger\right] ,$$

where a_k and $a_k^\dagger$ commute with $\hat{x}_\mu$ and define oscillators: $\left[a_k, a_k^\dagger\right] = 2\omega(k)\delta(k-k')$.

QFT.....:

◇ The "free" field $\hat{\Phi}$ "coinciding with the Heisenberg field initially" after time translation by amount τ using the free Schrödinger Hamiltonian $\hat{H}_0 = \int \frac{dk}{2\omega(k)} a_k^\dagger a_k$, becomes

$$U_0(\tau)\left(\hat{\Phi}\right) = e^{i\tau \hat{H}_0} \hat{\Phi} e^{-i\tau \hat{H}_0} \ ,$$

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- The interaction Hamiltonian is accordingly

$$\hat{H}_I\left(x_0\right) = \lambda : S_{x_0}\left(U_0(\tau)\left(\hat{\Phi}\right)^4\right) : = \lambda : S_{x_0+\tau}\left(\hat{\Phi}^4\right) : , \lambda > 0 \ ,$$

where $:\ :$ denotes the normal ordering of a_k and $a_k^\dagger$.

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- The S-matrix S can be worked out.

Quantised evolutions...

- We can easily extend our earliar presentations to the cylinder, a version related to 2+1 gravity and $\mathbb{R} \times S^3$. In all these models, only discrete time translations are possible, a result known before.

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- One striking consequence of quantised time translations is that even though a time independent Hamiltonian is an observable, in scattering processes, it is conserved only modulo $\frac{2\pi}{\theta}$, where θ is the noncommutative parameter.
- Scattering theory is also formulated and an approach to quantum field theory is outlined hep-th/0410067,JHEP 0411(2004) 068.

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- Scattering theory is also formulated and an approach to quantum field theory is outlined hep-th/0410067,JHEP 0411(2004) 068.
- But for the moment we will concentrate on only the noncommutative cylinder.

Noncommutative cylinder $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$

◇ It is generated by $\hat{x}_0$ and $e^{-i\hat{x}_1}$ with the relation

$$\left[\hat{x}_0\,,\,e^{-i\hat{x}_1}\right]=\theta\,e^{-i\hat{x}_1}\,.$$

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- For $\theta=0$, there is a close relation between $C^\infty\left(\mathbb{R}\times\mathbb{R}\right)$ and the functions $C^\infty\left(\mathbb{R}\times S^1\right)$ on a cylinder. The former is generated by coordinate functions $\hat{x}_0$ and $\hat{x}_1$, and the latter by $\hat{x}_0$ and $e^{i\hat{x}_1}$, $e^{i\hat{x}_1}$ being invariant under the 2π-shifts $\hat{x}_1\to\hat{x}_1\pm 2\pi$.

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- Following this idea, we can regard the noncommutative $\mathbb{R}\times S^1$ algebra $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ as generated by $\hat{x}_0$ and $e^{i\hat{x}_1}$ with the defining relation $e^{i\hat{x}_1}\hat{x}_0=\hat{x}_0e^{i\hat{x}_1}+\theta e^{i\hat{x}_1}\,,$

Noncommutative cylinder $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$

- For the noncommutative cylinder we get:

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- so that for the spectrum spec $\hat{x}_0$ of $\hat{x}_0$ in an IRR, we have,

$$\text{spec}\,\hat{x}_0 = \theta\mathbb{Z} + \frac{\theta\varphi}{2\pi} = \theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \equiv \left\{\theta\left(n + \frac{\varphi}{2\pi}\right) : n \in \mathbb{Z}\right\}\,.$$

Noncommutative cylinder....

- We can realise $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ irreducibly in the auxiliary Hilbert space $L^2\left(S^1, dx_1\right)$. It has the scalar product given by

$$(\alpha,\beta)=\int_0^{2\pi} dx_1\,\alpha^*\left(e^{ix_1}\right)\beta\left(e^{ix_1}\right)\;,\;\alpha,\beta\in L^2\left(S^1,dx_1\right)\;.$$

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- On this space, $e^{i\hat{x}_1}$ acts by evaluation map,

$$\left(e^{i\hat{x}_1}\alpha\right)\left(e^{ix_1}\right)=e^{ix_1}\alpha\left(e^{ix_1}\right)\ ,$$

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while $\hat{x}_0/\theta$ acts like the $\theta=0$ momentum with domain $D_\varphi(\hat{p}_1)$.

- Now because of the spectral result, $e^{i\left(\omega+\frac{2\pi}{\theta}\right)\hat{x}_0}=e^{i\varphi}e^{i\omega\hat{x}_0}$

Noncommutative cylinder....

- Thus elements of $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$ are quasiperiodic in ω and we can restrict ω to its fundamental domain:

$$\omega \in \left[-\frac{\pi}{\theta}, \frac{\pi}{\theta}\right].$$

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◇ The general element of $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ is thus

$$\hat{\alpha} = \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}_n(\omega) e^{in\hat{x}_1} e^{i\omega\hat{x}_0} ,$$

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- The symbol of $\hat{\alpha}$ is a function α on $\theta\left(\mathbb{Z}+\frac{\varphi}{2\pi}\right)\times S^1$: $\alpha : \theta\left(\mathbb{Z}+\frac{\varphi}{2\pi}\right)\times S^1 \to \mathbb{C}$.

Positive map & innerproduct..

- and is defined by:

$$\alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right)=\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega\, \tilde{\alpha}_n(\omega) e^{inx_1} e^{i\omega\theta\left(m+\frac{\varphi}{2\pi}\right)}.$$

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◇ Positive map is $S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}$:

$$S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}(\hat{\alpha}) = \int_0^{2\pi} dx_1\, \alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) .$$

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◇ We then have, for inner product,

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- We can infer the spectrum of the momentum operator $\hat{P}_1$ when it acts on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$.

Momentum....

◇ For the construction of a Hilbert space, we do not need this algebra, it is enough to have an $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}\right)$ -module which can be consistently treated.

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$$\left\langle \hat{\gamma} = e^{i\frac{\psi}{2\pi}\hat{x}_1} \sum_{n\in\mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} d\omega \tilde{\gamma}_n(\omega) e^{in\hat{x}_1} e^{i\omega\hat{x}_0} \right\rangle .$$

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- Such a module is $\mathcal{A}_\theta\left(\mathbb{R}\times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) =$

$$\left\langle \hat{\gamma} = e^{i\frac{\psi}{2\pi}\hat{x}_1}\sum_{n\in\mathbb{Z}}\int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} d\omega\tilde{\gamma}_n(\omega)e^{in\hat{x}_1}e^{i\omega\hat{x}_0}\right\rangle .$$

- The eigenvalues of $\hat{P}_1$ are now shifted by $\frac{\psi}{2\pi}$:

$$\hat{P}_1 e^{i\frac{\psi}{2\pi}\hat{x}_1}e^{in\hat{x}_1} = \left(n+\frac{\psi}{2\pi}\right)e^{i\frac{\psi}{2\pi}\hat{x}_1}e^{in\hat{x}_1}\,,\; n\in\mathbb{Z}\,.$$

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The Schrödinger constraint

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is not quasiperiodic in ω, continuous time translations and the Schrödinger constraint in the original form cannot be defined on $\mathcal{A}_\theta\left(\mathbb{R}\times S^1\right)$.

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- But translation of $\hat{x}_0$ by $\pm\theta$ leaves its spectrum intact. Hence the conventional Schrödinger constraint is thus changed to a discrete Schrödinger constraint.

The Schrödinger constraint

◇ The family of vector states constrained by the discrete Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta\left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) =$$

$$\left\{\hat{\psi} \in \mathcal{A}_\theta\left(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}\right) : e^{-i\theta\left(i\partial_{x_0}\right)}\hat{\psi} = e^{-i\theta\hat{H}}\hat{\psi}\right\}.$$

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$$\hat{\psi} = e^{-i\hat{x}_0^R \hat{H}\left(e^{i\hat{x}_1^L}, \hat{P}_1\right)} e^{i\frac{\psi}{2\pi}\hat{x}_1} \hat{\chi}\left(e^{i\hat{x}_1}\right) ,$$

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- *See JHEP 11(2004)068 for further details*