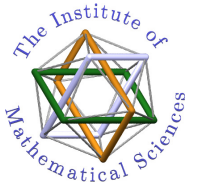


Space-time Noncommutativity and quantised evolution

T R Govindarajan, IMSC, Chennai, India

`trg@imsc.res.in`

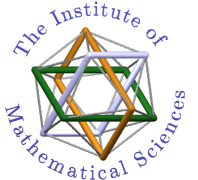
BRATISLAVA, June 2007



noncommutative spacetime & unitary ...

- ◇ We will start with $1 + 1$ dimensional theory. And look at the spacetime commutators of the form:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta\epsilon_{\mu\nu}\mathcal{I}$$

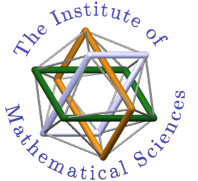


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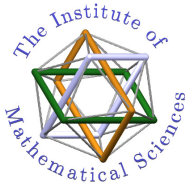


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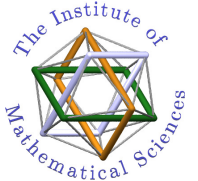
- ◇ Its usually remarked that this leads to non unitary quantum theory. We believe this is due to incorrect appreciation of the role of "Time".
- ◇ But the correct statement is if a group of transformations cannot be implemented on the algebra $\mathcal{A}_\theta(\mathcal{R}^2)$ generated by \hat{x}_μ with our relation then it will not be a symmetry Even this should be improved - will come back later



noncommutative spacetime ...

- ◇ We readily see that spacetime translations are automorphisms of $\mathcal{A}_\theta(\mathcal{R}^2)$: With $\mathcal{U}(\vec{a})\hat{x}_\mu = \hat{x}_\mu + a_\mu$ we see that,

$$[\mathcal{U}(\vec{a})\hat{x}_\mu, \mathcal{U}(\vec{a})\hat{x}_\nu] = i\theta\varepsilon_{\mu\nu} .$$



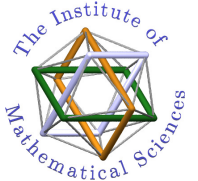
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$$U(\tau) := \mathcal{U}((\tau, 0))$$



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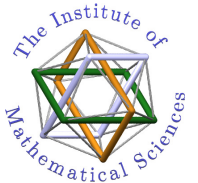
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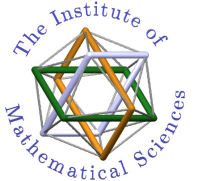
- ◇ Without the time-translation automorphism, we cannot formulate conventional quantum physics.



noncommutative spacetime..

- ◇ The infinitesimal generators of $\mathcal{U}(\vec{a})$ can be obtained from

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noncommutative spacetime..

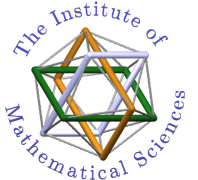
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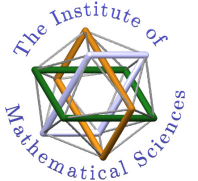
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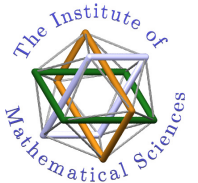
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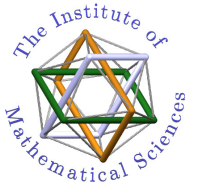
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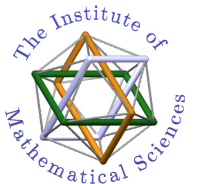
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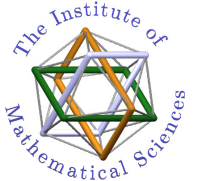
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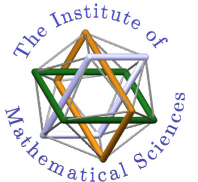
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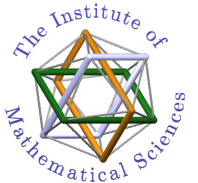
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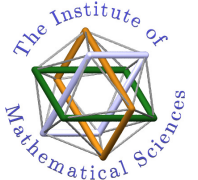
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- ◇ Causality: It is impossible to localize (the representation of) “coordinate” time \hat{x}_0 in $\mathcal{A}_\theta(\mathbb{R}^2)$ sharply. This leads to failure of causality Chaichian et al.



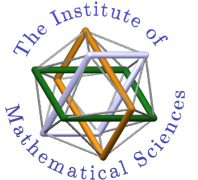
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- ◇ The following important point was emphasised to us by Doplicher. In quantum mechanics, if \hat{p} is momentum, $\exp(i\xi\hat{p})$ is spatial translation by amount ξ . This ξ is not the eigenvalue of the position operator \hat{x} . In the same way, the amount τ of time translation is not “coordinate time”, the eigenvalue of \hat{x}_0 . It makes sense to talk about a state and its translate by $U(\tau)$



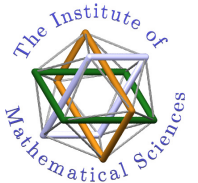
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- ◇ Concepts like duration of an experiment for $\theta = 0$ are expressed using $U(\tau)$. They carry over to the noncommutative case too.



Representation theory..

- ◇ Observables, states and dynamics of quantum theory are to be based on the algebra $\mathcal{A}_\theta(\mathbb{R}^2)$. Here we develop the formalism for their construction.

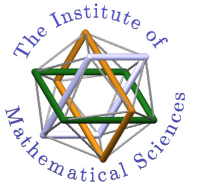


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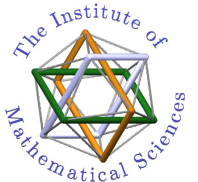
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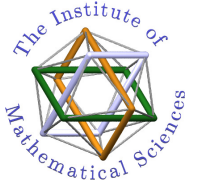
- ◇ An “inner” product on $\mathcal{A}_\theta(\mathbb{R}^2)$ is needed for an eventual construction of a Hilbert space.



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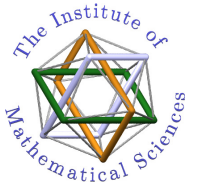
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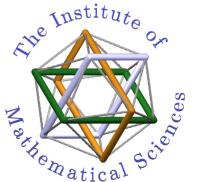
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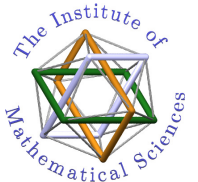
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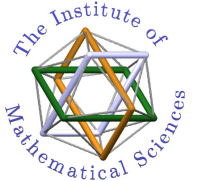
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- ◇ We illustrate these ideas briefly in the context of the commutative case, when $\theta = 0$



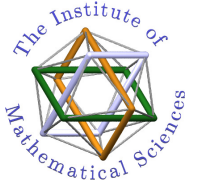
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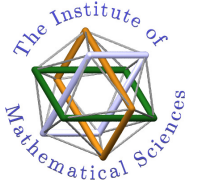
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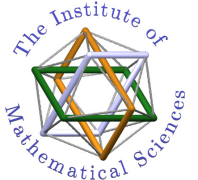
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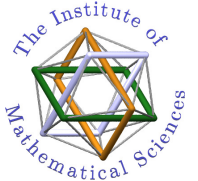
- ◇ We get a family of spaces \mathcal{C}_t with a positive-definite sesquilinear form $(\cdot, \cdot)_t$:

$$(\psi, \varphi)_t = \int dx_1 \psi^*(t, x_1) \varphi(t, x_1).$$



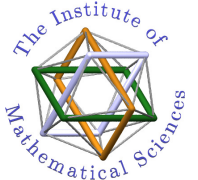
The commutative case

- ◇ Every function $\hat{\alpha}$ which vanishes at time t is a two-sided ideal $\mathcal{I}_t^{\theta=0} = \mathcal{I}_t^0$ of \mathcal{C} . As elements of \mathcal{C}_t , they become null vectors.



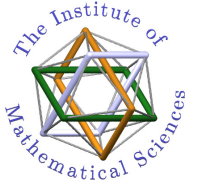
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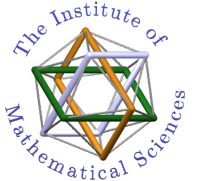
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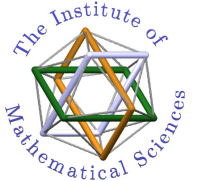
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- ◇ For elements $\psi + \mathcal{N}_t^0$ and $\chi + \mathcal{N}_t^0$ in $\mathcal{C}_t/\mathcal{N}_t^0$, the scalar product is

$$(\psi + \mathcal{N}_t^0, \chi + \mathcal{N}_t^0)_t = (\psi, \chi)_t .$$



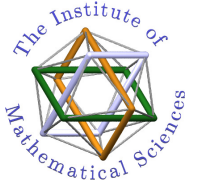
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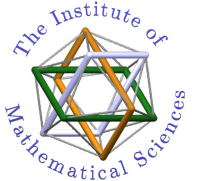
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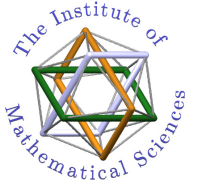
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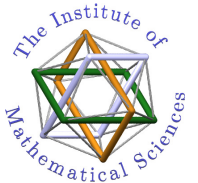
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- ◇ The subspace $\tilde{\mathcal{H}}_{0,t}$ depends on the Hamiltonian H and is chosen as follows.



The commutative case

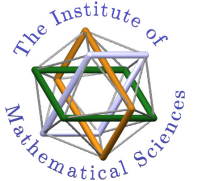
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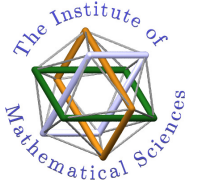
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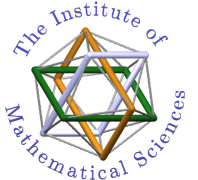
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- ◇ but on $\tilde{\mathcal{H}}_{0,t}$, it fulfills this property:

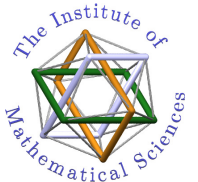


The commutative case

◇ We notice since,

$$\begin{aligned}\psi(x_0 + \tau, x_1) &= \left(e^{-i\tau(i\partial_{x_0})} \psi \right) (x_0, x_1) \\ &= \left(e^{-i\tau H} \psi \right) (x_0, x_1),\end{aligned}$$

time evolution preserves the norm of $\psi \in \tilde{\mathcal{H}}_{0,t}$.
Therefore if it vanishes at $x_0 = t$, it vanishes identically
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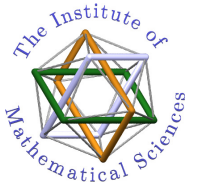
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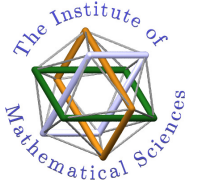
- ◇ The completion of $\tilde{\mathcal{H}}_{0,t}$ is the quantum Hilbert space \mathcal{H}_t^0 . There is no convenient inclusion of \mathcal{H}_t^0 in $\hat{\mathcal{H}}_t^0$.



The commutative case

- ◇ Under time evolution by amount τ , ψ becomes

$$e^{-i\tau H}\psi = e^{-i(\hat{x}_0 + \tau)H}\psi_0 \in \tilde{\mathcal{H}}_{0,t} .$$

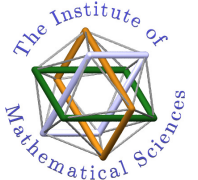


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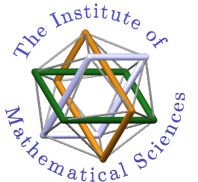
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- ◇ An observable \hat{K} has to respect the Schrödinger constraint and leave $\tilde{\mathcal{H}}_{0,t}$ (and hence \mathcal{H}_t^0) invariant. This means that

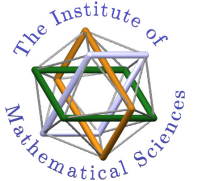
$$\left[i\partial_{x_0} - H, \hat{K} \right] = 0 .$$



The commutative case

- ◇ Under time translation, \hat{x}_0 in \hat{K} shifts to $\hat{x}_0 + \tau$ as it should:

$$\hat{K}(\tau) = e^{-i\tau H} \hat{K} e^{+i\tau H} = e^{-i(\hat{x}_0 + \tau)H} \hat{L} e^{+i(\hat{x}_0 + \tau)H} .$$



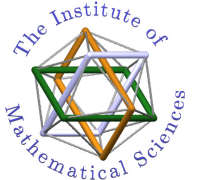
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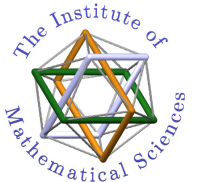
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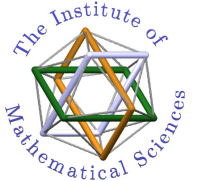
- What we have described above leads to conventional physics. As expected \hat{x}_0 is not an observable as it does not commute with $i\partial_{x_0} - H$:

$$[\hat{x}_0, i\partial_{x_0} - H] = -i\mathbb{I} .$$



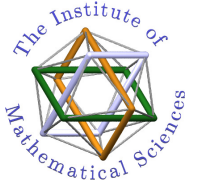
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- ◇ The above discussion shows that for quantum theory, what we need are: (1) a suitable inner product on $\mathcal{A}_\theta(\mathbb{R}^2)$; (2) a Schrödinger constraint on $\mathcal{A}_\theta(\mathbb{R}^2)$; and (3) a Hamiltonian \hat{H} and observables which act on the constrained subspace of $\mathcal{A}_\theta(\mathbb{R}^2)$.



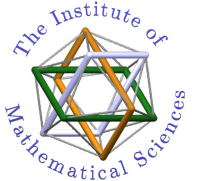
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- ◇ We now consider these one by one.

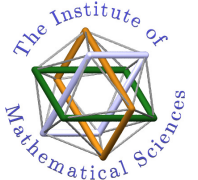


The symbol calculus

- ◇ The first inner product is based on symbol calculus. If $\hat{\alpha} \in \mathcal{A}_\theta(\mathbb{R}^2)$, we write it as

$$\hat{\alpha} = \int d^2k \tilde{\alpha}(k) e^{ik_1 \hat{x}_1} e^{ik_0 \hat{x}_0},$$

and associate the symbol α_S with $\hat{\alpha}$



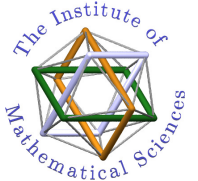
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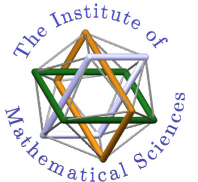
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- ◇ The symbol is a function on \mathbb{R}^2 . It is NOT the MOYAL symbol. Using this symbol, we can define a positive map S_t by

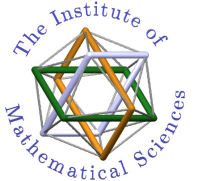
$$S_t(\hat{\alpha}) = \int dx_1 \alpha_S(t, x_1).$$



The Schrödinger constraint

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$$i\frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta}\text{ad } \hat{x}_1 ,$$



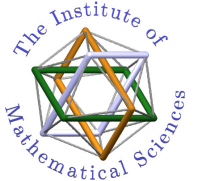
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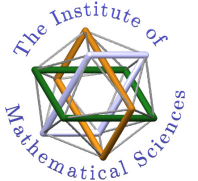
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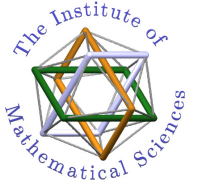
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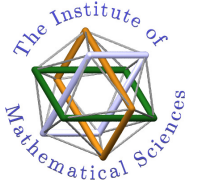
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- ◇ The states constrained by the Schrödinger equation is

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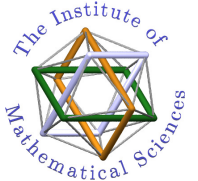
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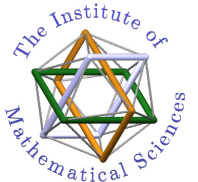
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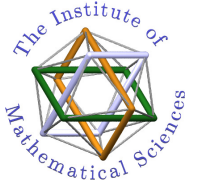
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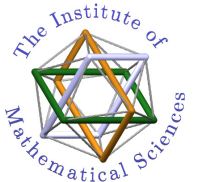
$$U(\hat{x}_0^R, \tau_I) = T \exp \left[-i \left(\int_{\tau_I}^{x_0} d\tau \hat{H}(\tau, \hat{x}_1^L, \hat{P}_1) \right) \right] \Big|_{x_0 = \hat{x}_0^R}$$



Some observations

- ◇ An alternative useful form for $\hat{\psi}$ is

$$\hat{\psi} = V \left(\hat{x}_0^R, -\infty \right) \hat{\chi} \left(\hat{x}_1 \right)$$



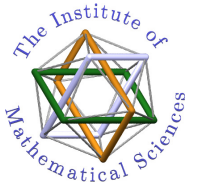
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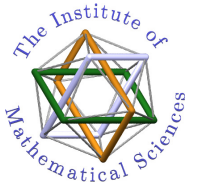
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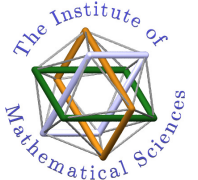
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- ◇ The Hilbert spaces \mathcal{H}_θ^S and \mathcal{H}_θ^V based on scalar products $(\cdot, \cdot)_S$ and $(\cdot, \cdot)_V$ are obtained from $\tilde{\mathcal{H}}_\theta$ by completion. Our basic assumption is that \hat{H} is self-adjoint in the chosen scalar product.



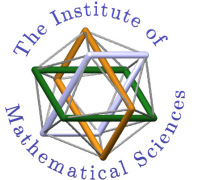
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- ◇ In the passage from H to \hat{H} , there is an apparent ambiguity. We replaced x_0 by \hat{x}_0^L , but we may be tempted to replace x_0 by \hat{x}_0^R . But it is incorrect to replace x_0 by \hat{x}_0^R and at the same time x_1 by \hat{x}_1^L . Time and space should NOT commute when θ becomes nonzero whereas \hat{x}_0^R and \hat{x}_1^L commute.



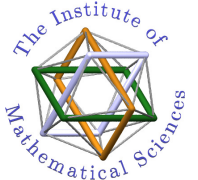
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- ◇ Note that $\hat{x}_0^L = -\theta\hat{P}_1 + \hat{x}_0^R$ and that \hat{x}_0^R behaves much like the $\theta = 0$ time x_0 . Thus if H has time-dependence, its effect on \hat{H} is to induce new momentum-dependent terms leading to nonlocal (“acausal”) interactions.



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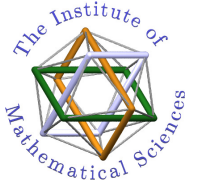
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- ◇ We can construct observables as before and no complications are encountered.



A spectral map:

- ◇ For $\theta = 0$ let the Hamiltonian be: $H = -\frac{1}{2m} \frac{\partial^2}{\partial x_1^2} + V(\hat{x}_1)$ with eigenstates ψ_E fulfilling the Schrödinger constraint:

$$\psi_E(\hat{x}_0, \hat{x}_1) = \varphi_E(\hat{x}_1) e^{-iE\hat{x}_0}, \quad H\varphi_E = E\varphi_E.$$



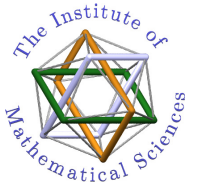
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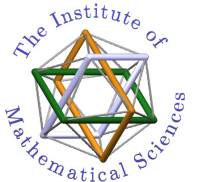
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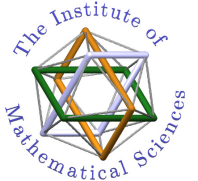
- ◇ Then \hat{H} has exactly the same spectrum as H and its eigenstates $\hat{\psi}_E$ are obtained from ψ_E .

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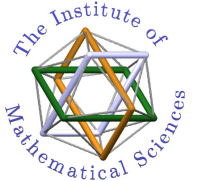
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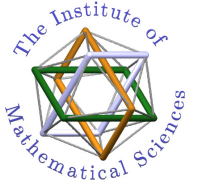
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- ◇ So for $\hat{\Phi}$, we write:

$$\hat{\Phi} = \int \frac{dk}{2\omega(k)} \left[a_k \hat{\phi}_k + a_k^\dagger \hat{\phi}_k^\dagger \right] ,$$

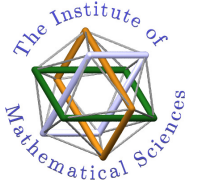
where a_k and a_k^\dagger commute with \hat{x}_μ and define oscillators: $\left[a_k, a_k^\dagger \right] = 2\omega(k)\delta(k - k')$.



QFT.....:

- ◇ The “free” field $\hat{\Phi}$ “coinciding with the Heisenberg field initially” after time translation by amount τ using the free Schrödinger Hamiltonian $\hat{H}_0 = \int \frac{dk}{2\omega(k)} a_k^\dagger a_k$, becomes

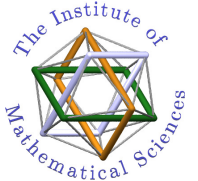
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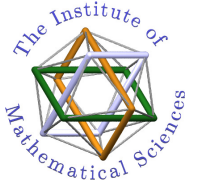
$$\hat{H}_I(x_0) = \lambda : S_{x_0} \left(U_0(\tau) \left(\hat{\Phi} \right)^4 \right) : = \lambda : S_{x_0+\tau} \left(\hat{\Phi}^4 \right) : , \lambda > 0 ,$$

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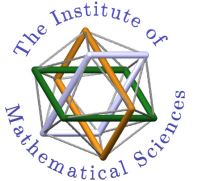
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- ◇ The S -matrix S can be worked out.

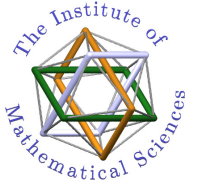
Quantised evolutions...

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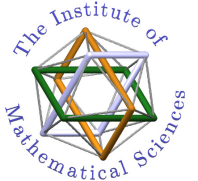
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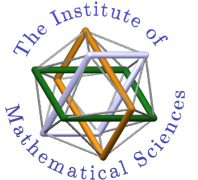
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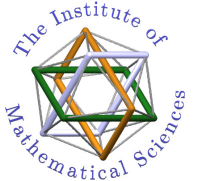
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- ◇ But for the moment we will concentrate on only the noncommutative cylinder.



Noncommutative cylinder $\mathcal{A}_\theta (\mathbb{R} \times S^1)$

- ◇ It is generated by \hat{x}_0 and $e^{-i\hat{x}_1}$ with the relation

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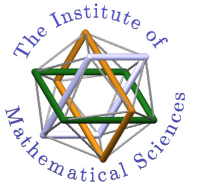


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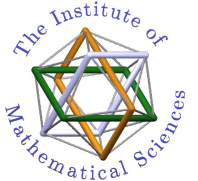


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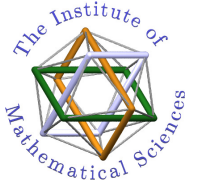
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- ◇ Following this idea, we can regard the noncommutative $\mathbb{R} \times S^1$ algebra $\mathcal{A}_\theta (\mathbb{R} \times S^1)$ as generated by \hat{x}_0 and $e^{i\hat{x}_1}$ with the defining relation $e^{i\hat{x}_1} \hat{x}_0 = \hat{x}_0 e^{i\hat{x}_1} + \theta e^{i\hat{x}_1}$,



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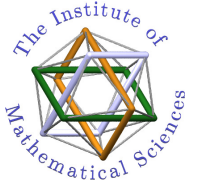
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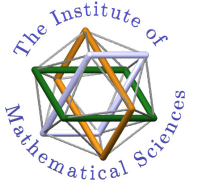
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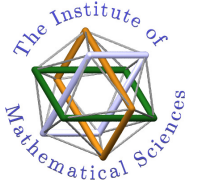
$$\text{spec } \hat{x}_0 = \theta\mathbb{Z} + \frac{\theta\varphi}{2\pi} = \theta \left(\mathbb{Z} + \frac{\varphi}{2\pi} \right) \equiv \left\{ \theta \left(n + \frac{\varphi}{2\pi} \right) : n \in \mathbb{Z} \right\} .$$



Noncommutative cylinder....

- ◇ We can realise $\mathcal{A}_\theta (\mathbb{R} \times S^1)$ irreducibly in the auxiliary Hilbert space $L^2 (S^1, dx_1)$. It has the scalar product given by

$$(\alpha, \beta) = \int_0^{2\pi} dx_1 \alpha^* (e^{ix_1}) \beta (e^{ix_1}) , \alpha, \beta \in L^2 (S^1, dx_1) .$$



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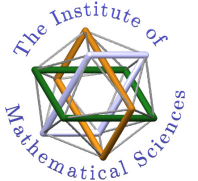
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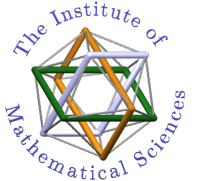
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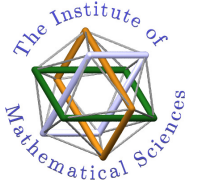
- ◇ Now because of the spectral result, $e^{i(\omega + \frac{2\pi}{\theta})\hat{x}_0} = e^{i\varphi} e^{i\omega\hat{x}_0}$



Noncommutative cylinder....

- ◇ Thus elements of $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ are quasiperiodic in ω and we can restrict ω to its fundamental domain:

$$\omega \in \left[-\frac{\pi}{\theta}, \frac{\pi}{\theta} \right].$$



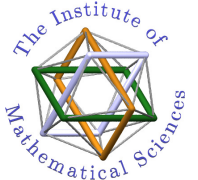
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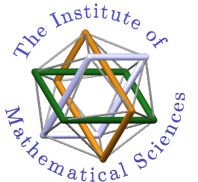
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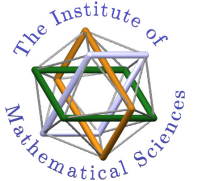
- ◇ The symbol of $\hat{\alpha}$ is a function α on $\theta (\mathbb{Z} + \frac{\varphi}{2\pi}) \times S^1$:
 $\alpha : \theta (\mathbb{Z} + \frac{\varphi}{2\pi}) \times S^1 \rightarrow \mathbb{C}.$



Positive map & innerproduct..

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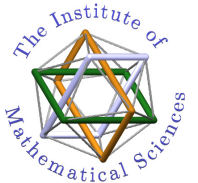


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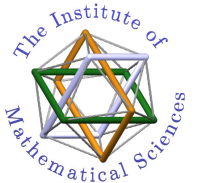
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◇ Positive map is $S_{\theta \left(m + \frac{\varphi}{2\pi} \right)}$:

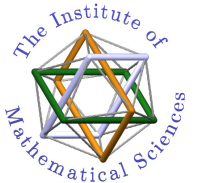
$$S_{\theta \left(m + \frac{\varphi}{2\pi} \right)} (\hat{\alpha}) = \int_0^{2\pi} dx_1 \alpha \left(\theta \left(m + \frac{\varphi}{2\pi} \right), e^{ix_1} \right) .$$



Positive map & innerproduct..

- ◇ We then have, for inner product,

$$\left(\hat{\alpha}, \hat{\beta}\right)_{\theta\left(m+\frac{\varphi}{2\pi}\right)} = S_{\theta\left(m+\frac{\varphi}{2\pi}\right)}\left(\hat{\alpha}^* \hat{\beta}\right)$$

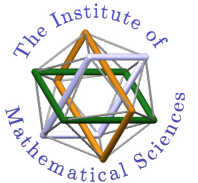


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- ◇ There are other possibilities for inner product such as the one based on coherent states. The equivalence of theories based on different inner products is discussed in our earlier work.

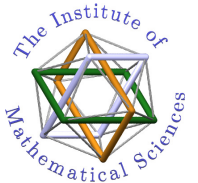


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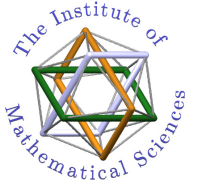
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- ◇ We can infer the spectrum of the momentum operator \hat{P}_1 when it acts on $\mathcal{A}_{\theta}\left(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}\right)$.



Momentum....

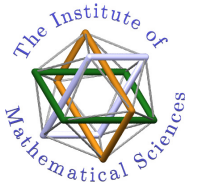
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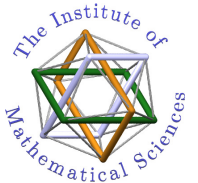
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◇ The eigenvalues of \hat{P}_1 are now shifted by $\frac{\psi}{2\pi}$:

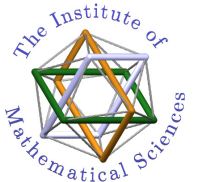
$$\hat{P}_1 e^{i\frac{\psi}{2\pi}\hat{x}_1} e^{in\hat{x}_1} = \left(n + \frac{\psi}{2\pi} \right) e^{i\frac{\psi}{2\pi}\hat{x}_1} e^{in\hat{x}_1}, \quad n \in \mathbb{Z}.$$



The Schrödinger constraint

- ◇ The inner product is still like

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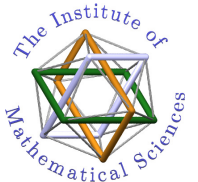
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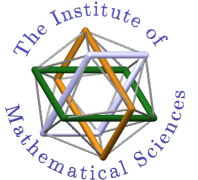
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- ◇ But translation of \hat{x}_0 by $\pm\theta$ leaves its spectrum intact. Hence the conventional Schrödinger constraint is thus changed to a discrete Schrödinger constraint.

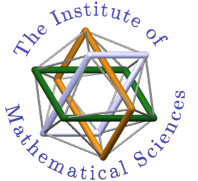


The Schrödinger constraint

- ◇ The family of vector states constrained by the discrete Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right) =$$

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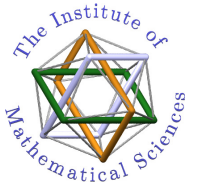
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- ◇ *See JHEP 11(2004)068 for further details*

